Transfer functions

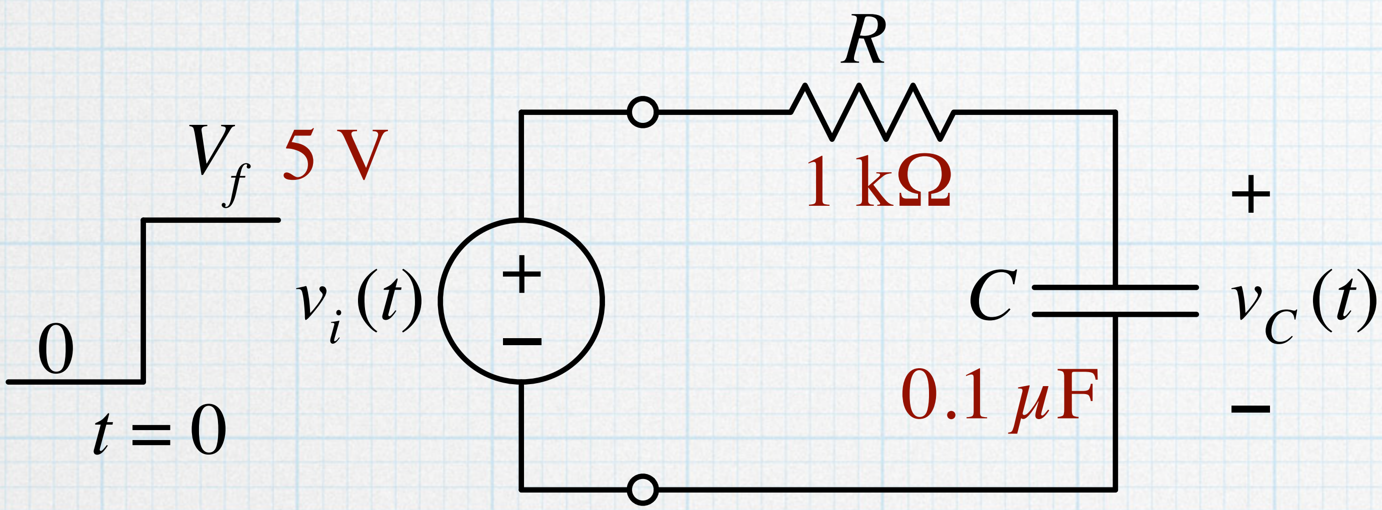
While there are many aspects to consider when determining the performance of circuit, two of the most common are:

1. Switching speed. Obviously, this is important in digital systems where where we would like logic circuits to switch as fast as possible. But switches are used in all sorts of applications. We can determine the switching speed of a circuit by looking at the step response.
2. Frequency response. This is essential in understanding how analog circuits change as the operating frequency changes. Frequency response is key to understanding the use of circuits in communications systems, audio systems, and control systems. Designing virtually any analog system will, at some level, depend on knowledge of how the circuits perform at different frequencies. Frequency response is usually determined by applying a sinusoid at the input and measuring the amplitude and phase shift at the output.

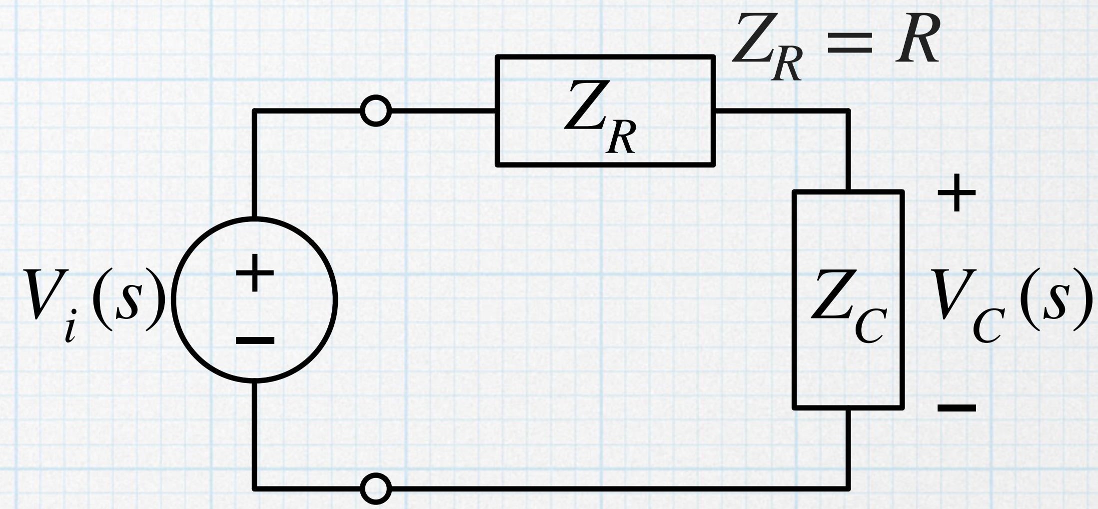
By now, after doing the examples in the previous sets of notes, it should be obvious that Laplace methods give us knowledge of both of these aspects of circuits. Take a circuit, describe in frequency-domain terms, apply a step source and a look at the output to determine switching speed. Take the same frequency-domain description of the circuit, apply an a sinusoid at the input, and look the output to determine the amplitude and phase shift.

Simple RC – switching speed

The simple *RC* circuit is rather mundane when looking at switching speed, but its simplicity makes it a good example to illustrate the idea.



$RC = 0.1$ msec



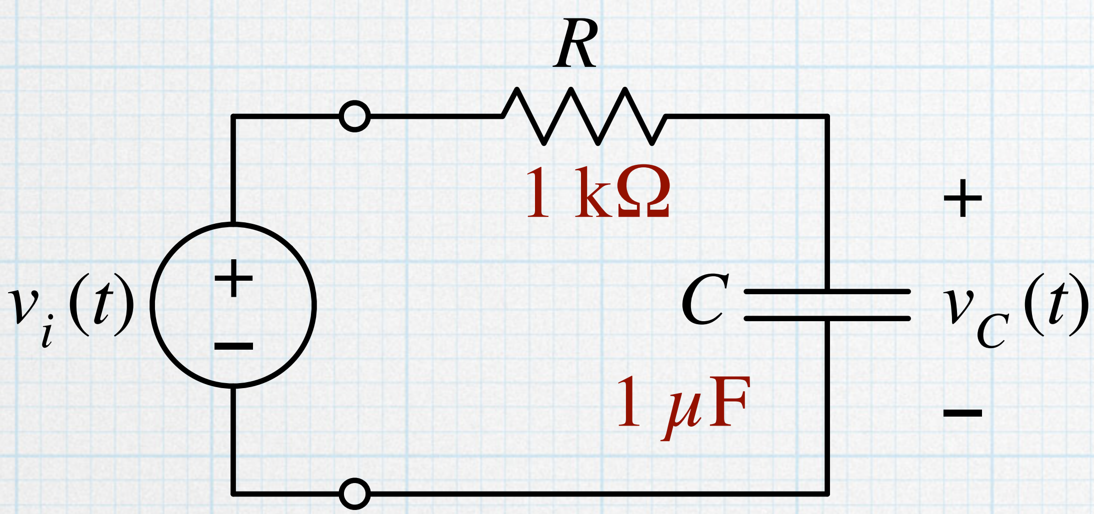
$$V_i(s) = \frac{V_f}{s} \qquad Z_C = \frac{1}{sC}$$

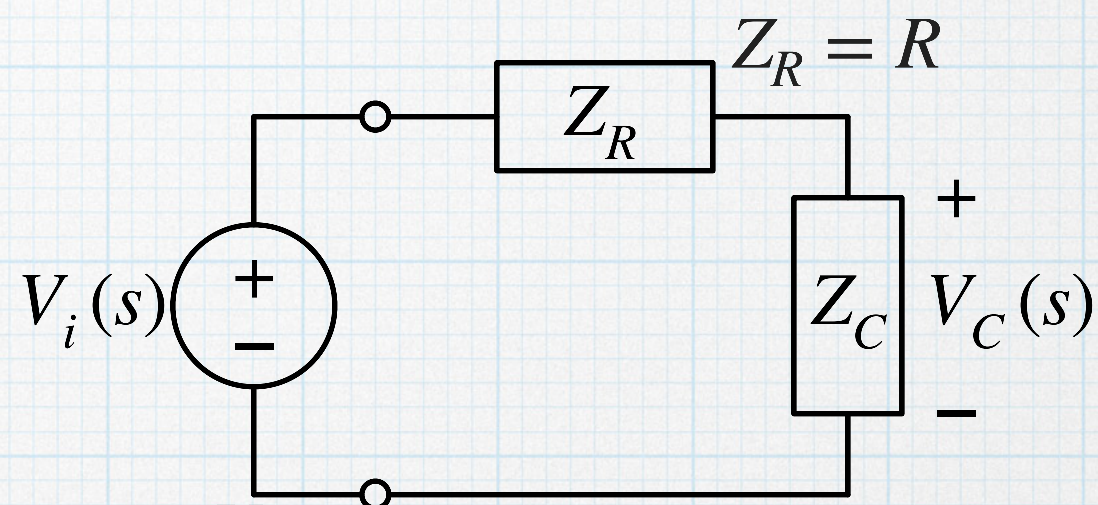
$$V_C(s) = \left(\frac{\frac{1}{RC}}{s + \frac{1}{RC}}\right)\left(\frac{V_f}{s}\right) = \frac{V_f/_{RC}}{s\left(s + \frac{1}{RC}\right)}$$

$$v_C(t) = V_f \cdot u(t) - V_f \cdot \exp\left(-\frac{t}{RC}\right) = (5\text{ V})\left[\mathrm{u}(t) - \exp\left(-\frac{t}{RC}\right)\right]$$

Simple RC – AC response

The *RC* circuit also exhibits a specific AC response. To find the response, look at the output of the *RC* circuit when driven by a sinusoidal source. From the earlier example:





$$v_i(t) = V_A \cdot \cos(\omega t)$$

$$V_A = 5\ \mathrm{V} \qquad \omega = 1000\ \mathrm{rad/s}$$

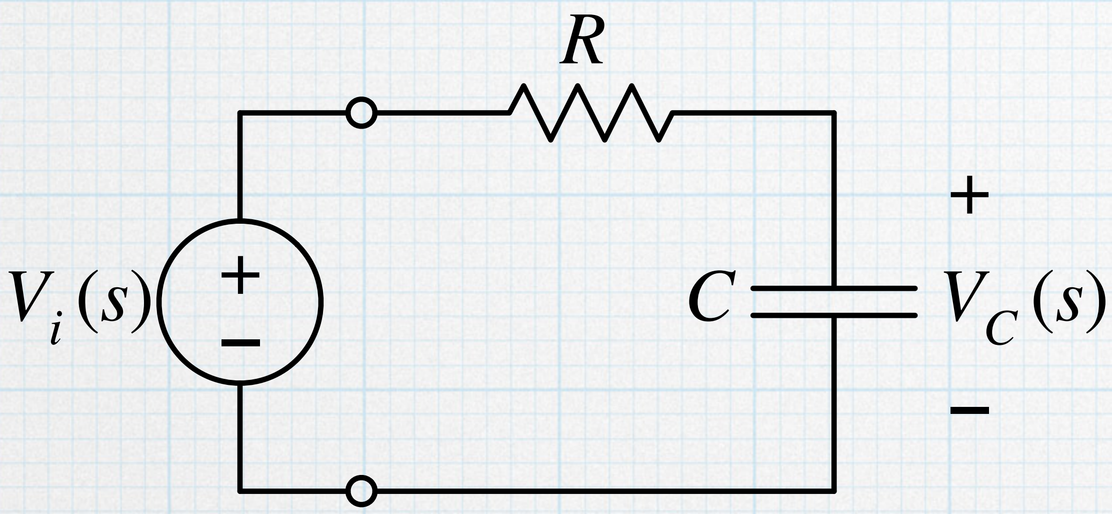
$$V_i(s) = \frac{V_A \cdot s}{s^2 + \omega^2} \qquad Z_C = \frac{1}{sC}$$

$$V_C(s) = \left(\frac{\frac{1}{RC}}{s + \frac{1}{RC}} \right) V_i(s) = \left(\frac{\frac{1}{RC}}{s + \frac{1}{RC}} \right) \left(\frac{V_A \cdot s}{s^2 + \omega^2} \right) = \frac{\frac{V_A}{RC} \cdot s}{\left(s + \frac{1}{RC} \right) \left(s^2 + \omega^2 \right)}$$

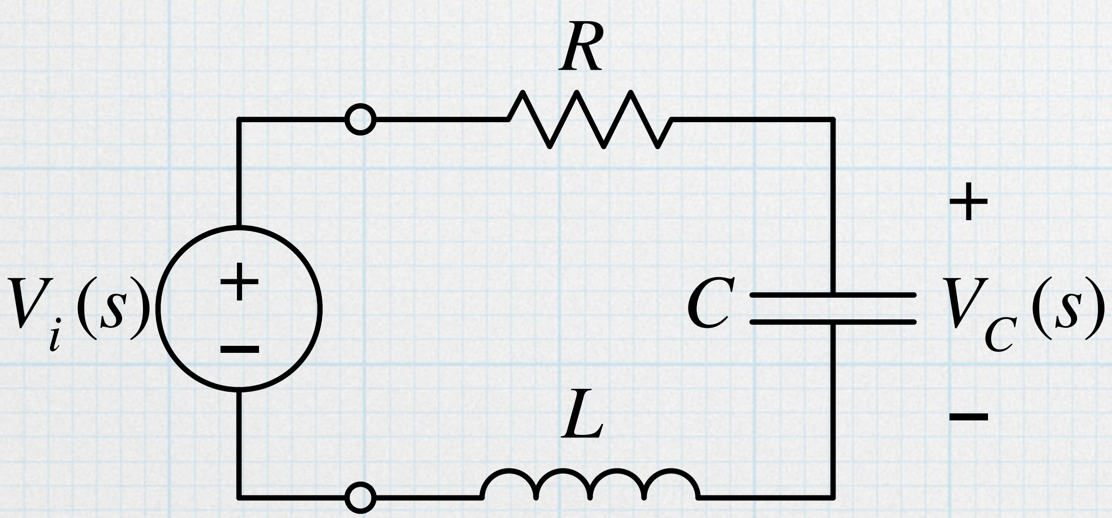
$$v_C(t) = V_1 \cdot e^{-\frac{t}{\tau}} + V_2 \cdot \cos(\omega t - \theta) \qquad \tau = \frac{1}{RC} = 1\ \mathrm{msec}$$

$$V_1 = \frac{-V_A}{1 + (\omega RC)^2} = -2.5\ \mathrm{V} \quad V_2 = \frac{V_A}{\sqrt{1 + (\omega RC)^2}} = 3.54\ \mathrm{V} \quad \theta = \arctan(\omega RC) = 45^\circ$$

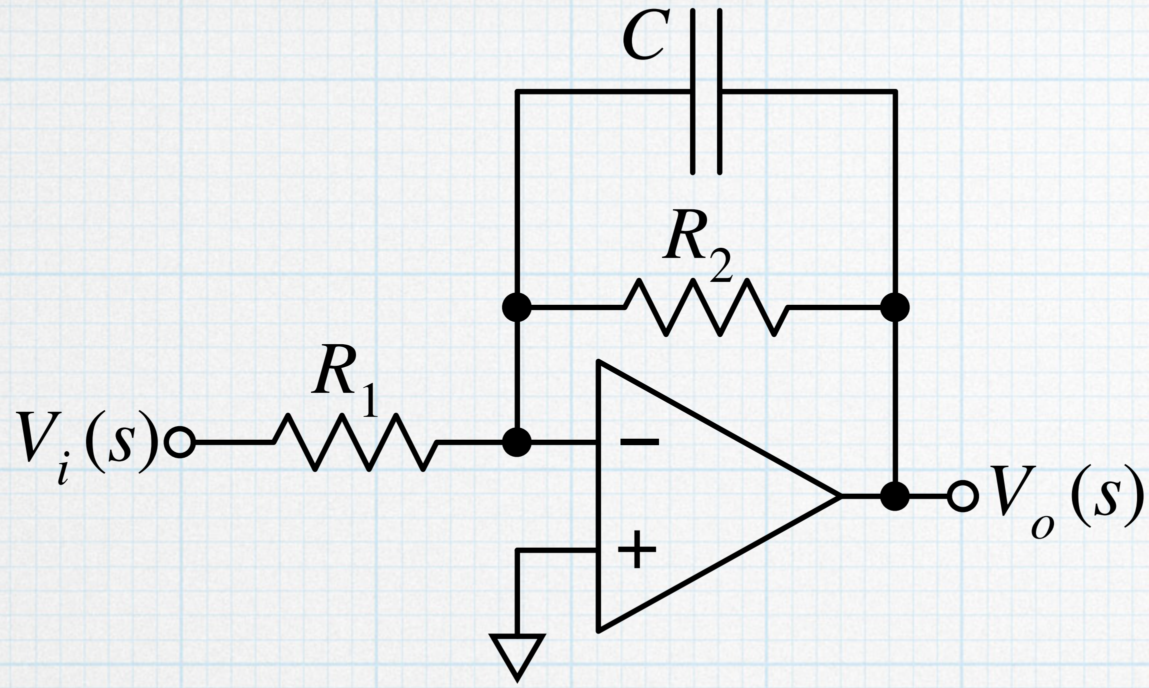
During the analysis each of the circuits in various examples, as we worked towards finding a specific voltage or current, we reached a stage where the quantity of interest was expressed as function multiplied by the source voltage (or current), $V_i(s)$.



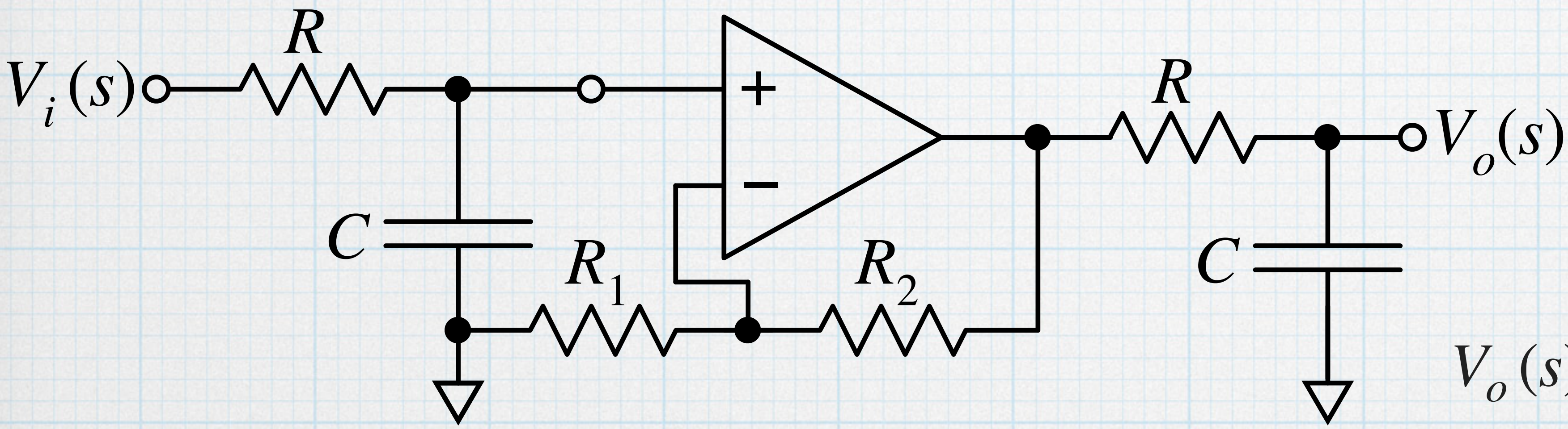
$$V_C(s) = \left(\frac{\frac{1}{RC}}{s + \frac{1}{RC}} \right) V_i(s)$$



$$V_C(s) = \left(\frac{\frac{1}{LC}}{s^2 + \frac{R}{L} + \frac{1}{LC}} \right) V_i(s)$$



$$V_o(s) = \left(-\frac{\frac{1}{R_1 C}}{s + \frac{1}{R_2 C}} \right) V_i(s)$$



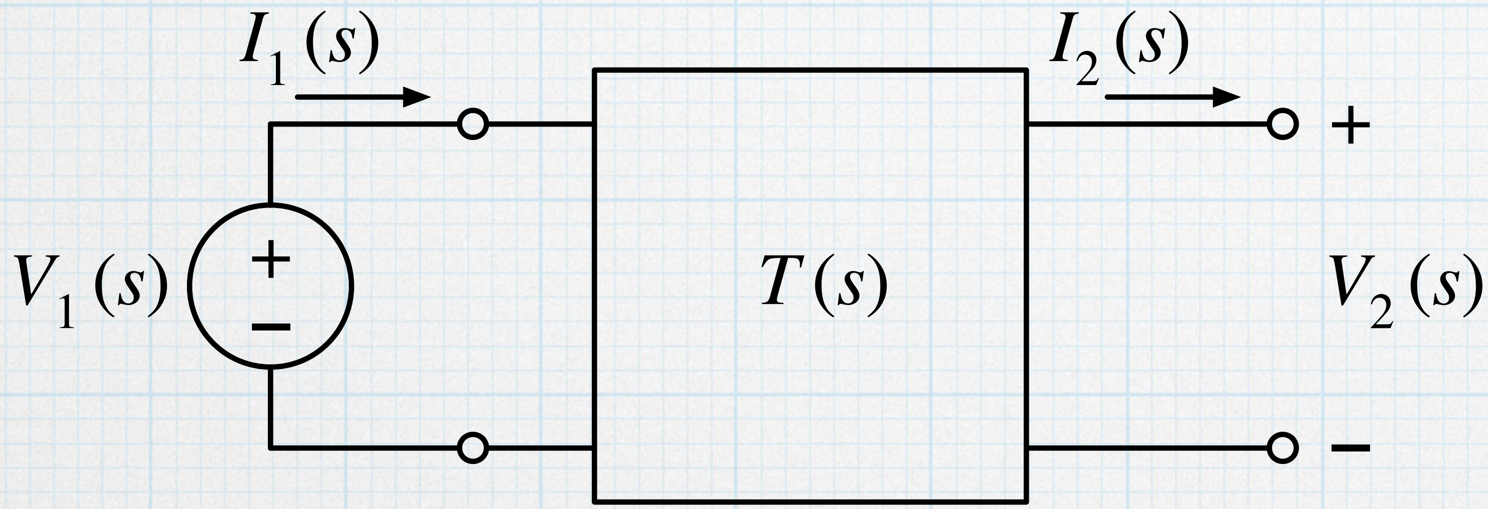
$$V_o(s) = \left[\frac{\left(1 + \frac{R_2}{R_1}\right)\left(\frac{1}{RC}\right)^2}{\left(s + \frac{1}{RC}\right)^2} \right] V_i(s)$$

We could show more examples, and they would reinforce the pattern that should be clear — when finding a voltage or current in the frequency domain, the result will always be in the form of a function that is derived from the circuit, multiplied by the Laplace transform of the source.

The frequency-domain expressions in the previous circuit examples could be generalized to a simple form:

$$V_2(s) = T(s) \cdot V_1(s)$$

The quantity relating the "output" to the "input" is known as the *transfer function*. The circuits could all be generalized to a simple block diagram:



There is an obvious similarity to two-port models studied in EE 201.

The transfer function depends only on the arrangement of the components in the circuit. It does not depend on the form of the source. The transfer function contains all of the information about the circuit — it determines the switching speed of the circuit and it determines the frequency response of the circuit. Those two behaviors are inextricably intertwined — if you know one, you can determine the other, although the math to do that could be messy. The key point is that everything depends on $T(s)$.

Often, a transfer function is a ratio of an "output" voltage to an "input" voltage.

$$T(s) = \frac{V_o(s)}{V_i(s)}$$

In this case, the transfer function is dimensionless and looks much like the gain that we have used for op amp circuits.

But a transfer function can be a ratio of currents, $I_o(s)/I_i(s)$ (also dimensionless), a ratio of a voltage to a current, $V_o(s)/I_i(s)$ (units of Ω), or the inverse — a ratio of current to voltage, $I_o(s)/V_i(s)$ (units of Ω^{-1} or siemens).

In working various examples, we have seen that frequency-domain quantities can always be expressed as a ratio of two polynomials.

$$T(s) = \frac{c_m s^m + c_{m-1} s^{m-1} + \ldots + c_1 s + c_o}{d_n s^n + d_{n-1} s^{n-1} + \ldots + d_1 s + d_o}$$

As we do more work with transfer functions, we will see that is convenient to put them into standard form. In particular, we like to have the leading coefficients of the numerator and denominator be equal to 1. It is easy to factor out the leading coefficients and put them into a constant "gain" factor.

$$T(s) = G_o \cdot \frac{s^m + a_{m-1} s^{m-1} + \ldots + a_1 s + a_o}{s^n + b_{n-1} s^{n-1} + \ldots + b_1 s + b_o}$$

Poles and Zeros

The polynomials in the numerator and denominator can be factored.

$$T(s) = G_o \cdot \frac{(s+Z_m)(s+Z_{m-1})\ldots(s+Z_1)(s+Z_o)}{(s+P_n)(s+P_{n-1})\ldots(s+P_1)(s+P_o)}$$

The various values of Z_m are the zeros of the numerator and hence are the zeros of the transfer function, $T(s = -Z_m) = 0$.

The various values of P_n are the zeros of the denominator and are the *poles* of the transfer function, $T(s = -P_n) \rightarrow \infty$.

The poles determine the basic behavior of the circuit. To see this, recall the partial fraction expansion of the function.

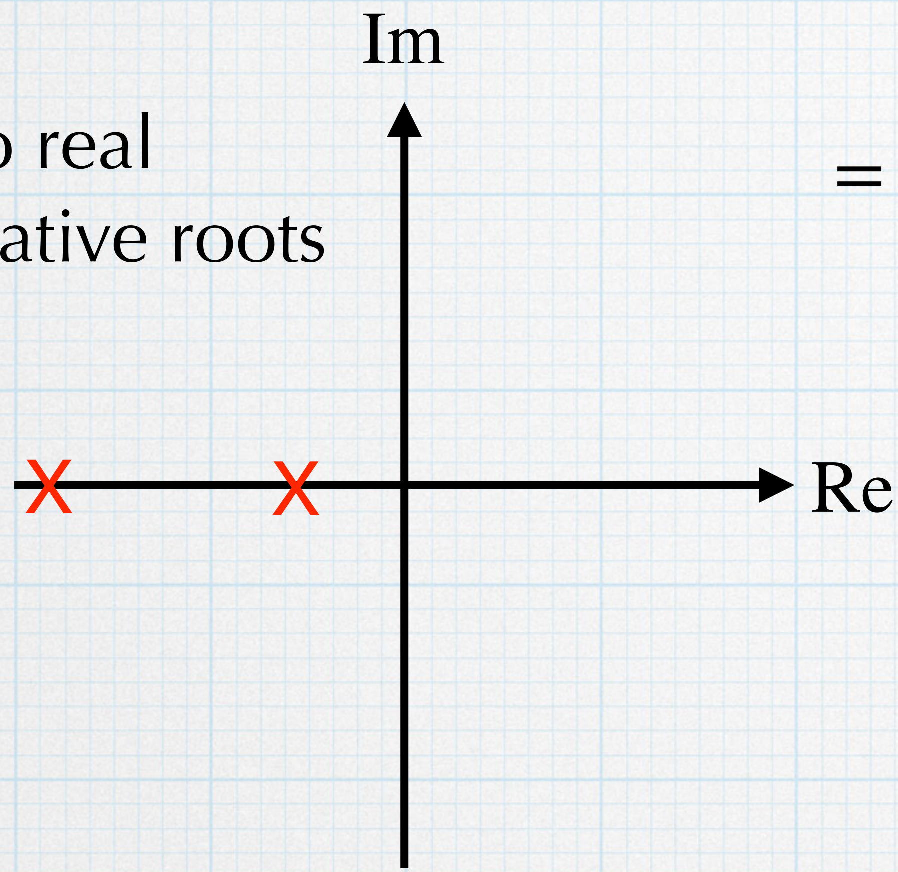
$$\frac{(s+Z_m)(s+Z_{m-1})\ldots(s+Z_1)(s+Z_o)}{(s+P_n)(s+P_{n-1})\ldots(s+P_1)(s+P_o)} =$$

$$\frac{A_n}{(s+P_n)} + \frac{A_{n-1}}{(s+P_{n-1})} + \ldots + \frac{A_1}{(s+P_1)} + \frac{A_o}{(s+P_o)}$$

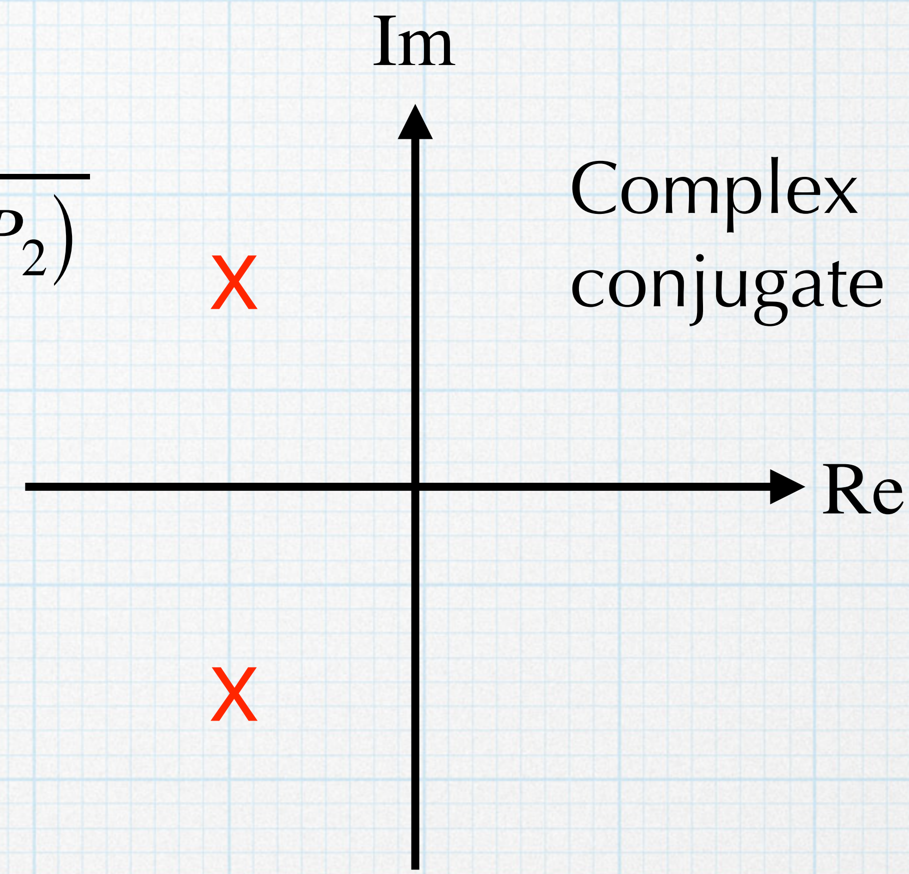
$$T(s) = \frac{1}{s^2 + as + b}$$

$$= \frac{1}{(s + P_1)(s + P_2)}$$

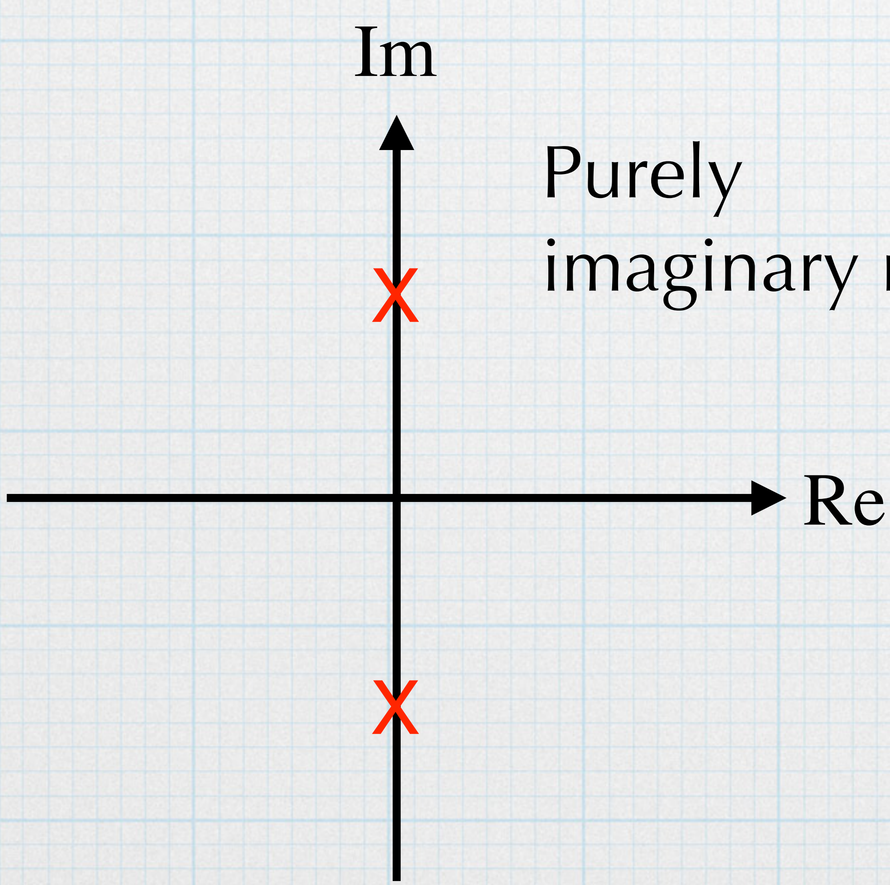
Two real negative roots



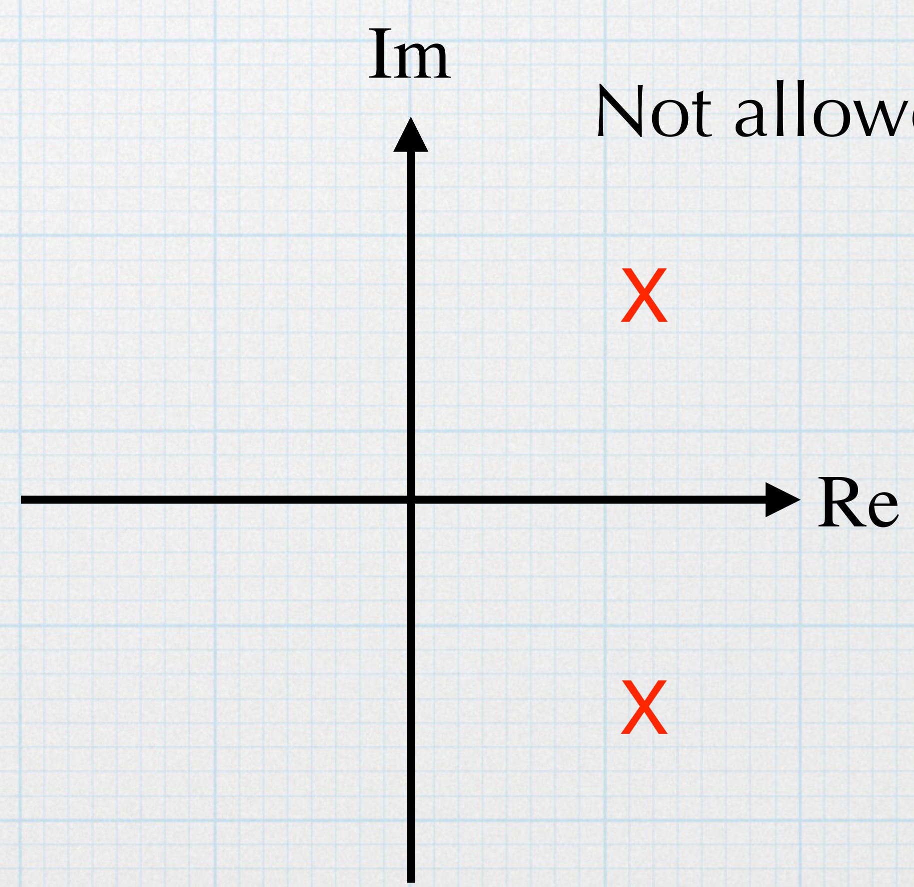
Complex conjugate roots



Purely imaginary roots



Not allowed!



Transfer functions from circuits made with passive components — resistors, capacitors, and inductors exhibit certain characteristics.

- At most, the numerator can be one order higher than the denominator. $m \leq n + 1$. Usually, the numerator has lower order than the denominator.
- Complex zeros or poles must come in the form of complex conjugates. (All of the coefficients in the polynomials must be real numbers.)
- Poles cannot be located in the right-half plane. ($\sigma > 0$)

Consider a circuit that has a second 2nd-order transfer function

$$T(s) = \frac{1}{s^2 + as + b} = \frac{1}{\left(s + P_1\right)\left(s + P_2\right)}$$

Drive it with a step function, $V_i(\mathrm{s}) = V_f/\mathrm{s}$. In the frequency domain, the output will have the form:

$$V_o(s) = T(s)\, V_i(s)$$

$$= \frac{V_f}{s\left(s^2 + as + b\right)}$$

$$= \frac{V_f}{s\left(s + P_1\right)\left(s + P_2\right)}$$

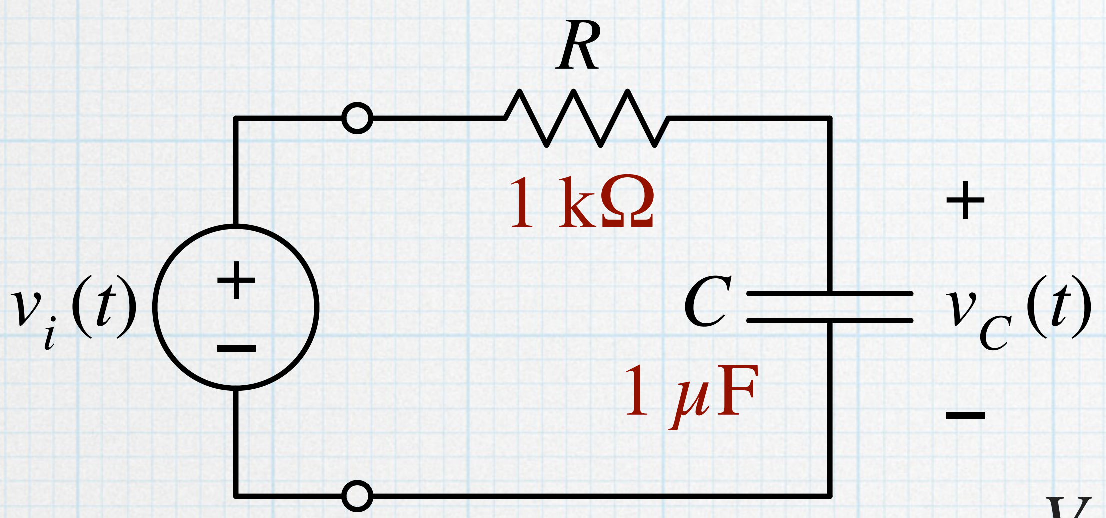
$$= \frac{V_1}{s} + \frac{V_2}{s + P_1} + \frac{V_3}{s + P_2}$$

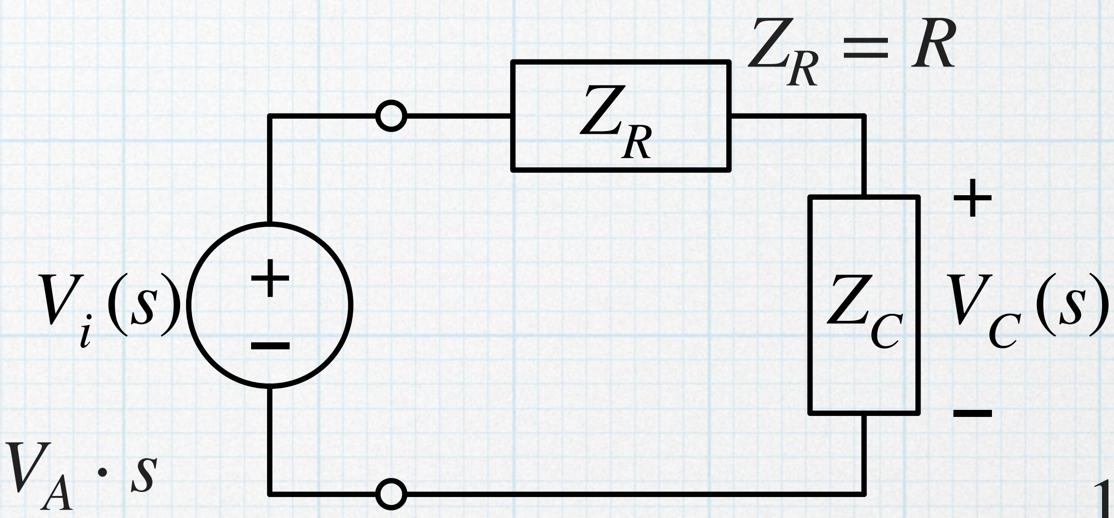
$$v_o(t) = V_1 \cdot u(t) + V_2 \exp\left(-P_1 t\right) + V_3 \exp\left(-P_2 t\right)$$

The complete response will have at least term that comes from the source (the step function in this case) and transient terms from the transfer function.

Frequency response

By using a sinusoidal source, we can look at a circuit's frequency response. Consider again the simple *RC* divider with an AC source.





$$V_i(s) = \frac{V_A \cdot s}{s^2 + \omega^2}$$

$$Z_C = \frac{1}{sC}$$

$$T(s) = \left(\frac{\frac{1}{RC}}{s + \frac{1}{RC}} \right)$$

$$V_C(s) = T(s)\,V_i(s) = \left(\frac{\frac{1}{RC}}{s + \frac{1}{RC}} \right) \left(\frac{V_A \cdot s}{s^2 + \omega^2} \right) = \frac{\frac{V_A}{RC} \cdot s}{\left(s + \frac{1}{RC} \right) \left(s^2 + \omega^2 \right)}$$

$$v_C(t) = V_1 \cdot e^{-\frac{t}{\tau}} + V_2 \cdot \cos(\omega t - \theta)$$

$$V_1 = \frac{-V_A}{1 + (\omega RC)^2} \qquad \tau = \frac{1}{RC} \qquad V_2 = \frac{V_A}{\sqrt{1 + (\omega RC)^2}} \qquad \theta = \arctan(\omega RC)$$

Very often, we are interested in how a voltage or current changes when the frequency of the sinusoid changes. In those cases, we are usually interested in *only* the sinusoidal part of the response — the transient portion is of secondary interest. So the first thing we do is to drop off the transient part of the solution and focus on the *steady-state* response of the system. (Recall sinusoidal steady-state analysis from EE 201.)

Another view of this approach is to say that we are restricting the complex frequency. By choosing to ignore the transient portion of the response, we are, in effect, setting $\sigma = 0$, so that $s = j\omega$ only.

After dropping the transient term, the steady-state response in the time domain is:

$$v_C(t) = V_2 \cdot \cos(\omega t - \theta) \qquad V_2 = \frac{V_A}{\sqrt{1 + (\omega RC)^2}} \qquad \theta = \arctan(\omega RC)$$

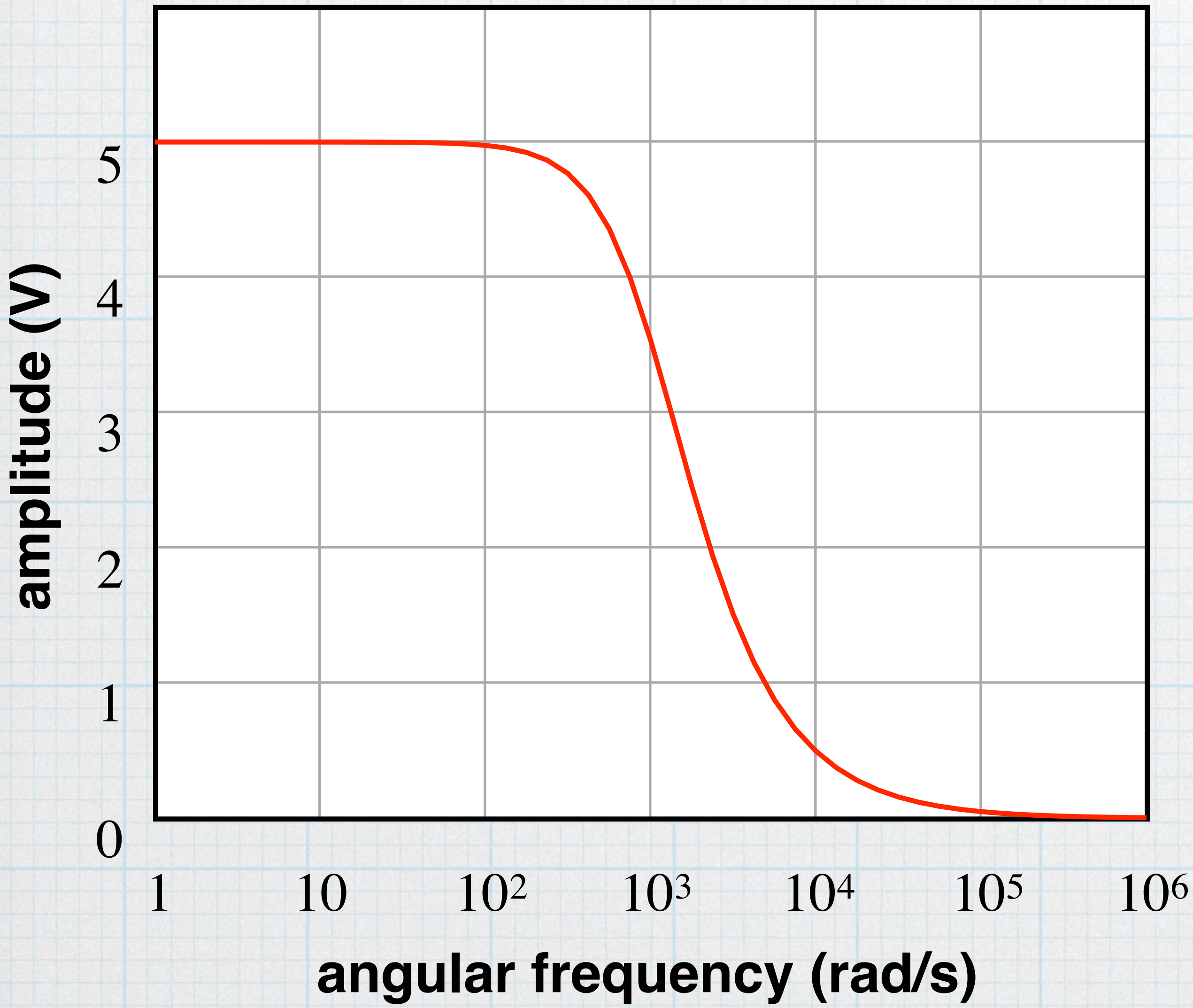
If the input voltage is a cosine $v_i(t) = V_A \cdot \cos(\omega t)$, then the steady-state output is also a cosine, but with a different amplitude and a phase shift. These are the key features when looking at frequency response. We know that the output will be a sinusoid, but we need to find the amplitude and phase.

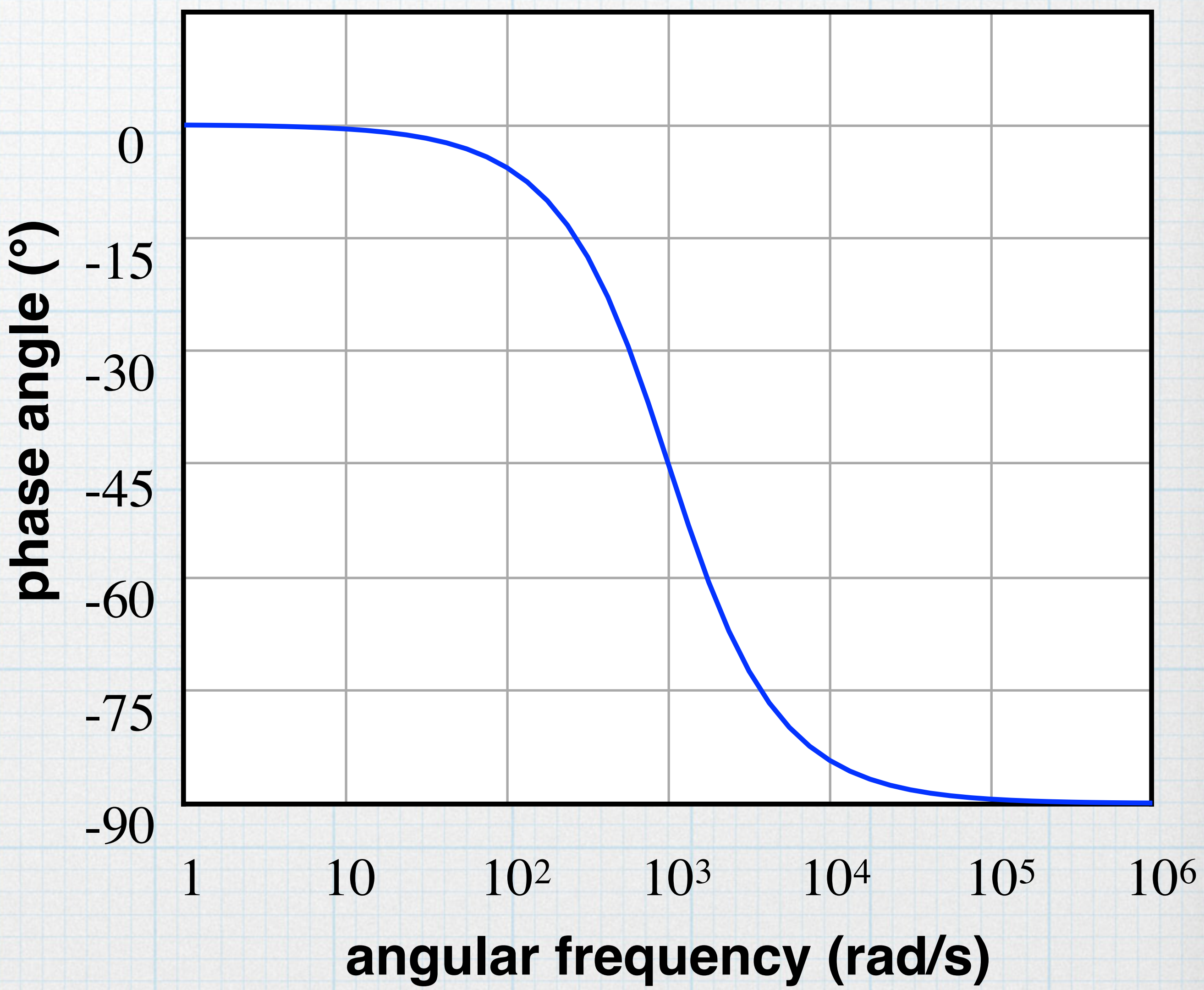
Frequency response plots

We can plot the amplitude and phase. Note: Because frequencies can range over several orders of magnitude, we almost always use a logarithmic scale for the frequency axis.

$$V_2 = \frac{V_A}{\sqrt{1 + (\omega RC)^2}}$$

$$\theta = \arctan(\omega RC)$$





The plots bear out the features of the amplitude and phase functions.

At low frequencies, $\omega \rightarrow 0$:

$$V_2 = \frac{V_A}{\sqrt{1 + (\omega RC)^2}} \rightarrow V_2 \approx V_A \qquad \theta = \arctan(\omega RC) \rightarrow \theta \approx 0^\circ$$

At high frequencies, $\omega RC >> 1$:

$$V_2 = \frac{V_A}{\sqrt{1 + (\omega RC)^2}} \rightarrow V_2 \approx \frac{V_A}{\omega RC} \qquad \theta = \arctan(\omega RC) \rightarrow \theta \approx -90^\circ$$

In between, when $\omega RC = 1$:

$$V_2 = \frac{V_A}{\sqrt{1 + (1)^2}} \rightarrow V_2 = \frac{V_A}{\sqrt{2}} \qquad \theta = \arctan(1) \rightarrow \theta = -45^\circ$$

Low frequencies "pass through" — the amplitude of the output is the about the same input — and high frequencies are attenuated — the amplitude of the output is decreasing inversely with frequency. We might call this a "low-pass" response. (More on this later.)

At this point, we have a method for finding the frequency response for a voltage or current in a circuit. Say that we want to find some the frequency response of some particular voltage, v_x, in a circuit.

1. Find the transfer function, $T(s) = V_x(s)/V_i(s)$.
2. Multiply the transfer function by the Laplace transform of a sinusoidal source, $V_x(s) = T(s)\cdot V_i(s)$, where $V_i(s)$ is the transform of a sine, cosine, or complex exponential.
3. Take the inverse transform to find corresponding time-domain function $V_x(s) \rightarrow v_x(t)$.
4. Drop off the transient portion of $v_x(t)$. Of course, the remaining steady-state part will be a sinusoidal function of some sort.
5. Use trigonometry and other math to convert the sinusoids into a single cosine term with amplitude and phase.
6. Use the amplitude and phase expressions to analyze the frequency response. Make a plot if desired.

But, wow, this whole approach seems clumsy. Is there more efficient method to find the amplitude and phase expressions?

The answer is “yes”. In fact, we will be able to do everything in the frequency domain — it will not be necessary to transform back. We will need to use some complex math, which we are usually willing to do if it saves time and effort in other places. The trick is to ignore all of the transients at the outset. To see how it works, suppose that we have a circuit characterized by the transfer function $T(s)$. We want to obtain the magnitude and phase of the output when we drive the circuit with a sinusoid. To keep it simple, use a complex exponential for the source.

$$v_i(t) = V_A \exp\left(j\omega t\right) \rightarrow V_i(s) = \frac{V_A}{s - j\omega}$$

In the frequency domain, the output will be

$$V_o(s) = T(s)\frac{V_A}{s - j\omega}$$

Without knowing the details of $T(s)$, it looks like we are limited in how far we can go. However, we know that if we do a partial fraction expansion, the result will have one term involving $s - j\omega$ along with some number of terms involving the poles of $T(s)$, whatever they may be.

$$V_o(s) = T(s)\frac{V_A}{s - j\omega} = \frac{A_o}{s - j\omega} + \frac{A_1}{s + P_1} + \frac{A_2}{s + P_2} + \cdots$$

$$V_o(s) = T(s)\frac{V_A}{s - j\omega} = \frac{A_o}{s - j\omega} + \frac{A_1}{s + P_1} + \frac{A_2}{s + P_2} + \cdots$$

All of the terms relating to the poles of $T(s)$ will be transient terms that will decay away over some period of time, even the complex conjugate poles. None of the terms relating to poles P_1, P_2, … will contribute to the steady-state response. The only term that matters for steady-state is the first one. So we need to calculate only A_o — the rest of the terms are irrelevant to our goal of finding the frequency response.

Proceeding as we have done with previous partial fraction expansions, we multiply both sides of the above equation, but this time we multiply only by $(s - j\omega)$.

$$T(s)\,V_A = A_o + \frac{A_1\left(s - j\omega\right)}{s + P_1} + \frac{A_2\left(s - j\omega\right)}{s + P_2} + \cdots$$

And then evaluate at $s = j\omega$. The result is

$$A_o = T\left(j\omega\right)V_A$$

The A_o coefficient is a complex number, because $T(j\omega)$ is a complex. We can express A_o in terms of magnitude and phase.

$$\left|A_o\right| e^{j\theta_A} = V_A \left|T\left(j\omega\right)\right| e^{j\theta_T}$$

The steady-state response in the time domain is the inverse transform

$$V_o(s) = \frac{\left|A_o\right| e^{j\theta_A}}{1 - j\omega} \rightarrow v_o(t) = \left|A_o\right| \exp\left[j\left(\omega t + \theta_A\right)\right]$$

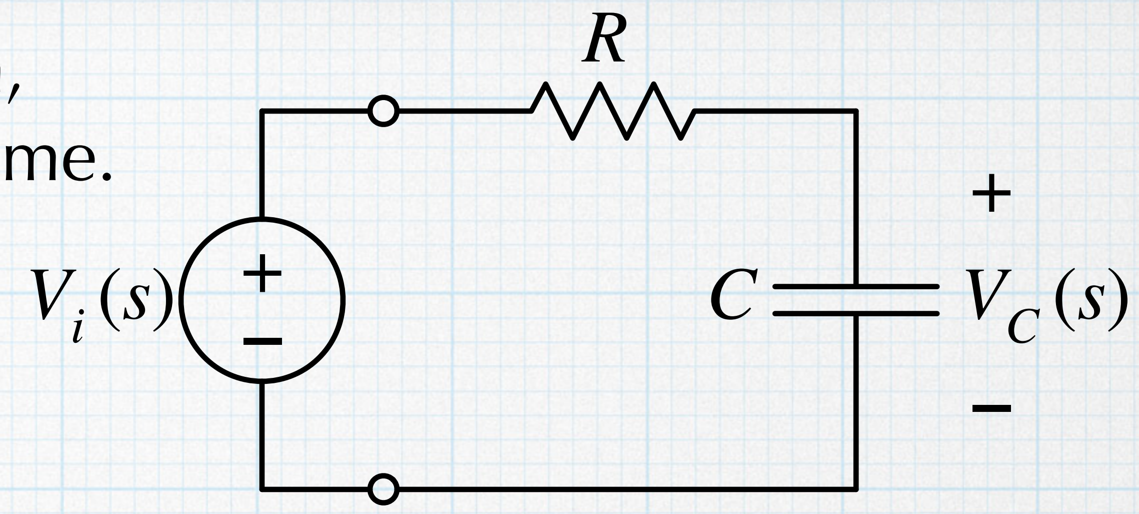
So by applying a sinusoid with amplitude V_A and frequency ω as the driving function to the circuit, the steady-state response is a sinusoid at with the same frequency and with amplitude of $|A_o| = |T(j\omega)| \cdot V_A$ and a phase shift of $\theta_A = \theta_T$. The transfer function, when evaluated at $s = j\omega$, has all of the pertinent information about the steady-state frequency response.

So the method for finding frequency response simplifies to:

1. Find the transfer function for the relevant quantity in the circuit.
2. Extract the magnitude and phase from the transfer function.
3. (If needed) Multiply the magnitude by the amplitude of the source and add the phase of the source to the phase of the transfer function.

Usually, step 3 is not necessary because the amplitude of the source is just a constant and the phase of the source is usually defined to be 0. The frequency dependence of the voltage or current is the same as the frequency dependance of the transfer function.

To illustrate the simplicity of the approach, analyze the RC divider circuit one more time. As we have seen, the transfer function is



$$T(s) = \left(\frac{\frac{1}{RC}}{s + \frac{1}{RC}}\right) = \frac{\omega_o}{s + \omega_o}$$

Substitute $s = j\omega$:

$$T(j\omega) = \left(\frac{\frac{1}{RC}}{j\omega + \frac{1}{RC}}\right) = \frac{\omega_o}{\omega_o + j\omega}$$

(Note that ω_o is a constant and ω is the frequency variable. Sometimes it can be confusing.)

Find the magnitude:

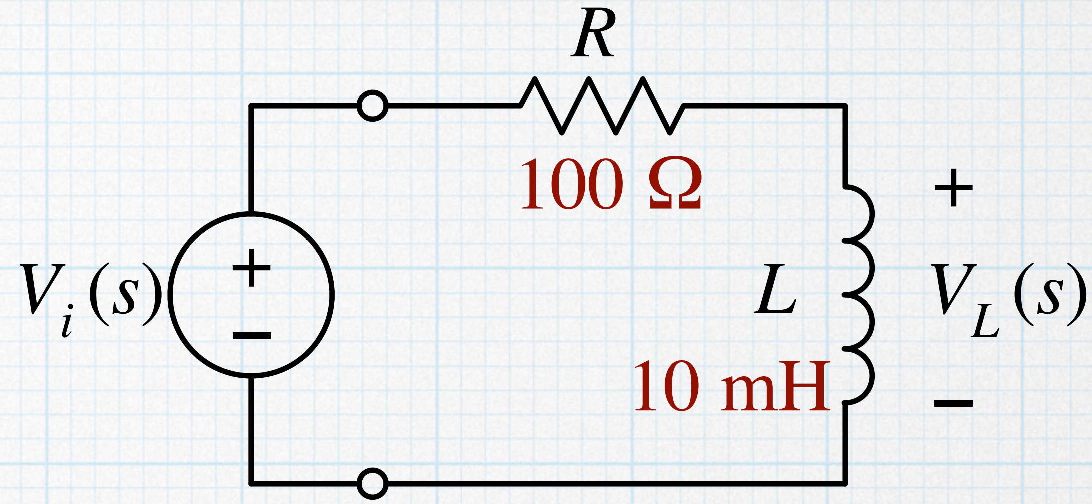
$$\left|T(j\omega)\right| = \frac{\omega_o}{\sqrt{\omega_o^2 + \omega^2}} = \frac{\frac{1}{RC}}{\sqrt{\left(\frac{1}{RC}\right)^2 + \omega^2}} = \frac{1}{\sqrt{1 + (\omega RC)^2}}$$

Multiply by the amplitude of the source, if desired. Then find the phase:

$$\theta_T = 0 - \arctan\left(\frac{\omega}{\omega_o}\right) = -\arctan(\omega RC)$$

The results are identical to the previous analysis. We have already made the plots.

Example 1



$Z_L = sL$

Find the frequency response for the inductor voltage *RL* circuit shown.

Once again, this is a simple voltage divider:

$$V_L(s) = \left(\frac{sL}{sL + R} \right) V_i(s)$$

The transfer function is:

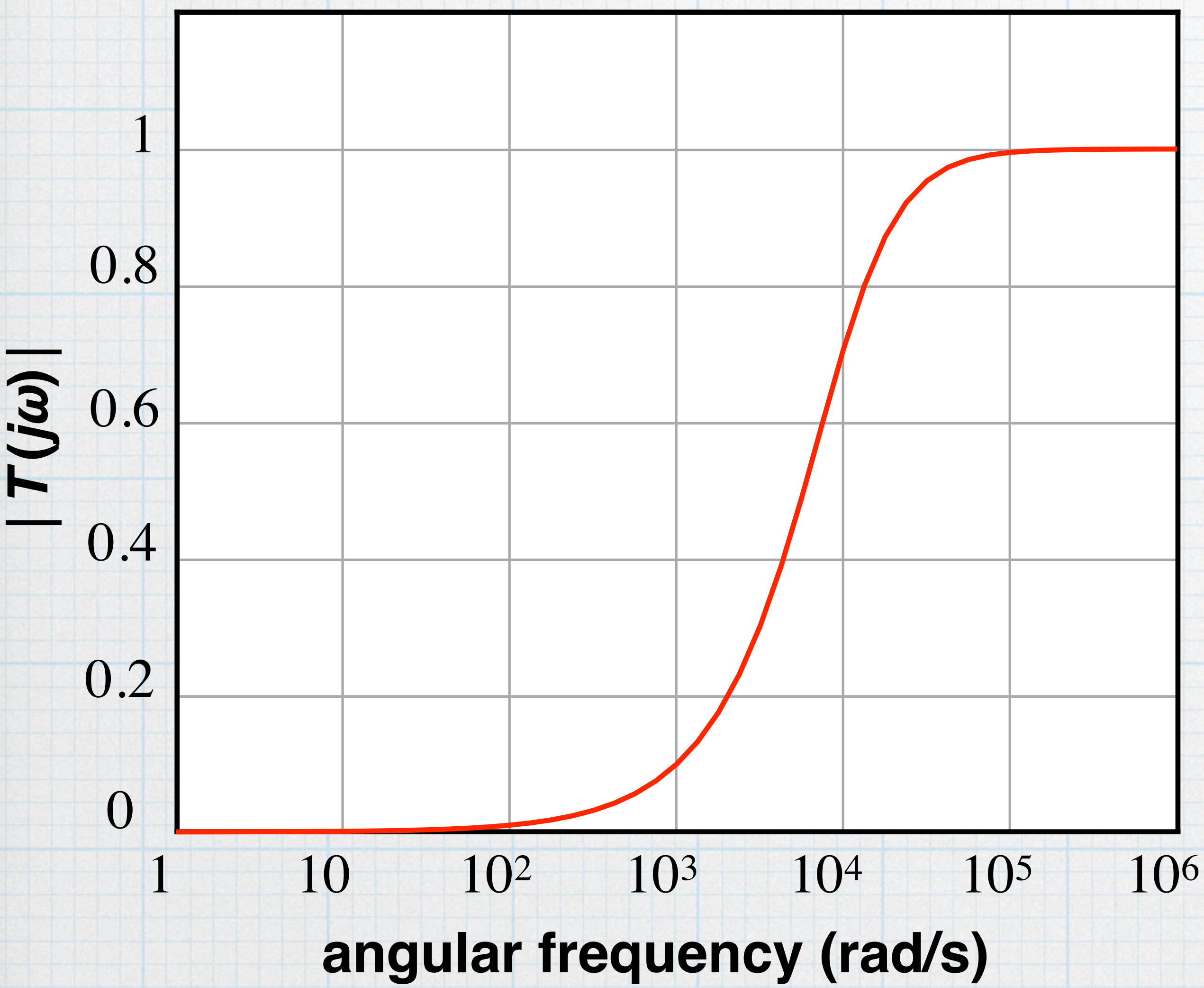
$$T(s) = \frac{V_L(s)}{V_i(s)} = \left(\frac{sL}{sL + R} \right) = \frac{s}{s + \frac{L}{R}} = \frac{s}{s + \omega_o} \qquad \omega_o = \frac{R}{L} = 10^4 \text{ rad/sec}$$

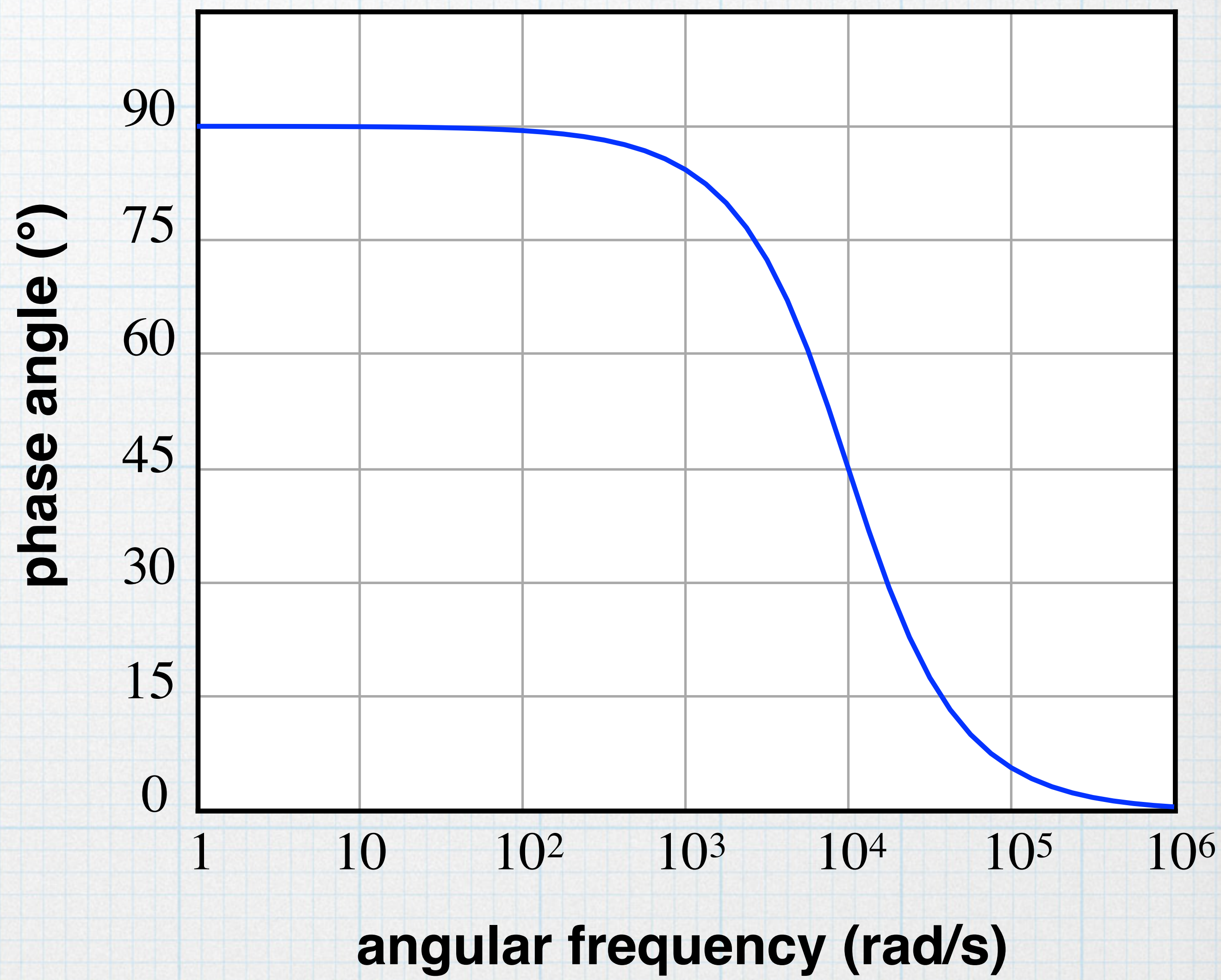
Substitute $s = j\omega$:

$$T(j\omega) = \frac{j\omega}{\omega_o + j\omega}$$

Magnitude: $\left|T(j\omega)\right| = \dfrac{\omega}{\sqrt{\omega_o^2 + \omega^2}} = \dfrac{\omega}{\sqrt{\left(\frac{R}{L}\right)^2 + \omega^2}} = \dfrac{1}{\sqrt{1 + \left(\frac{R}{\omega L}\right)^2}}$

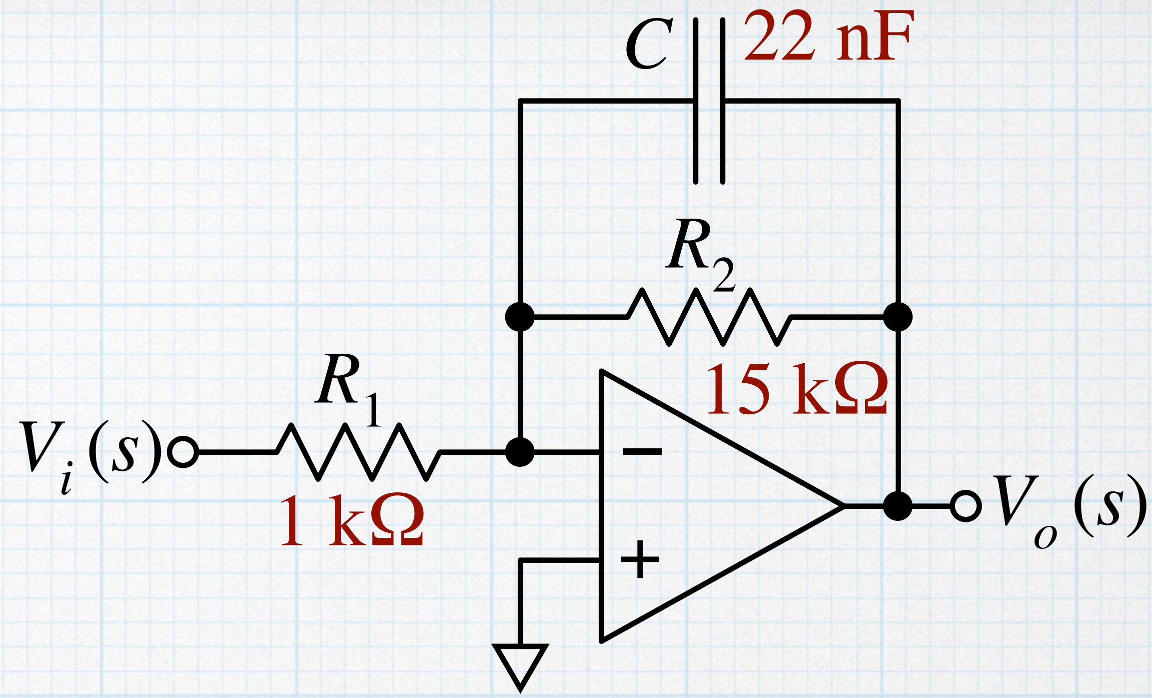
Phase: $\theta_T = 90° - \arctan\left(\dfrac{\omega}{\omega_o}\right) = 90° - \arctan\left(\dfrac{\omega L}{R}\right)$





This one has a different behavior. It starts low at low frequencies and the magnitude increases as the frequency increases.

Example 2



Find the frequency response for the amplifier circuit shown.

We have previously calculated the transfer function:

$$T(s) = \left(-\frac{\frac{1}{R_1 C}}{s + \frac{1}{R_2 C}} \right)$$

Rewriting slightly to emphasize the gain of the amp:

$$T(s) = \left(-\frac{R_2}{R_1} \right) \left(\frac{\frac{1}{R_2 C}}{s + \frac{1}{R_2 C}} \right) = G_o \frac{\omega_o}{s + \omega_o}$$

$$G_o = -\frac{R_2}{R_1} = -15$$

$$\omega_o = \frac{1}{R_2 C} = 3030 \text{ rad/sec}$$

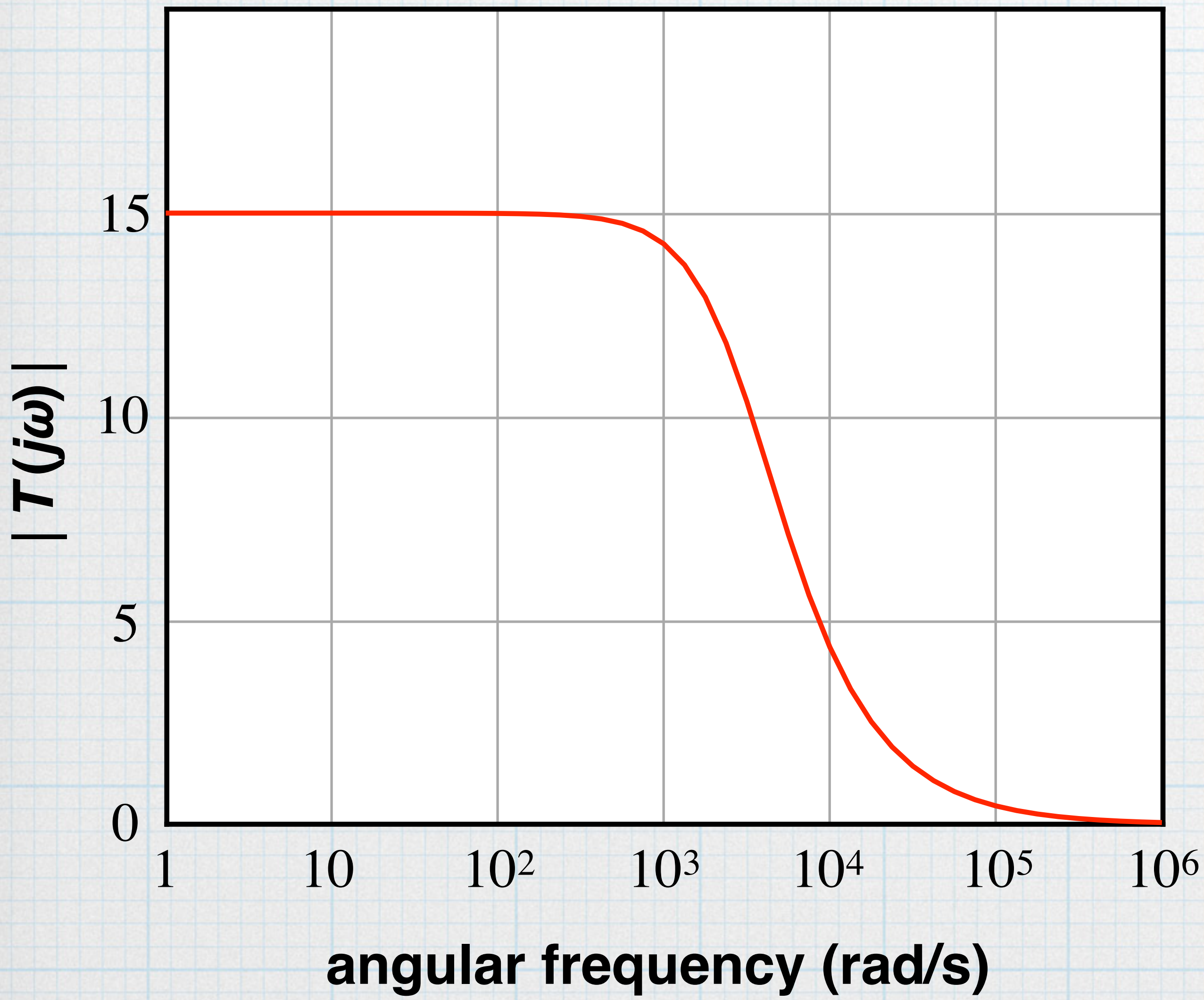
Substitute $s = j\omega$:

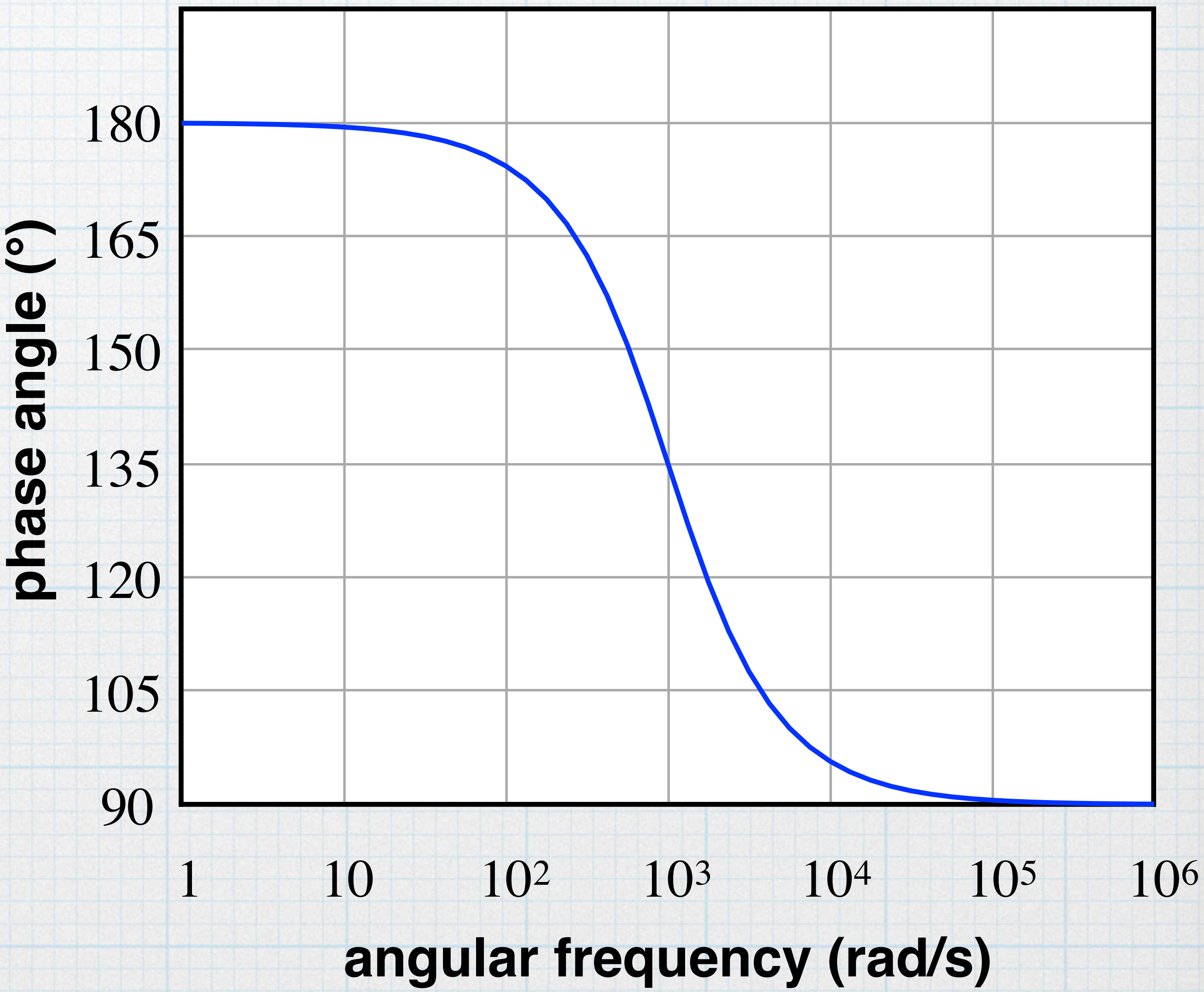
$$T(j\omega) = G_o \frac{\omega_o}{\omega_o + j\omega}$$

Magnitude: $\left|T(j\omega)\right| = \left|G_o\right| \dfrac{\omega_o}{\sqrt{\omega_o^2 + \omega^2}} = \dfrac{\left|G_o\right|}{\sqrt{1 + \omega R_2 C}}$

Phase: $\theta_T = 180^\circ - \arctan\left(\dfrac{\omega}{\omega_o}\right) = 180^\circ - \arctan\left(\omega RC\right)$

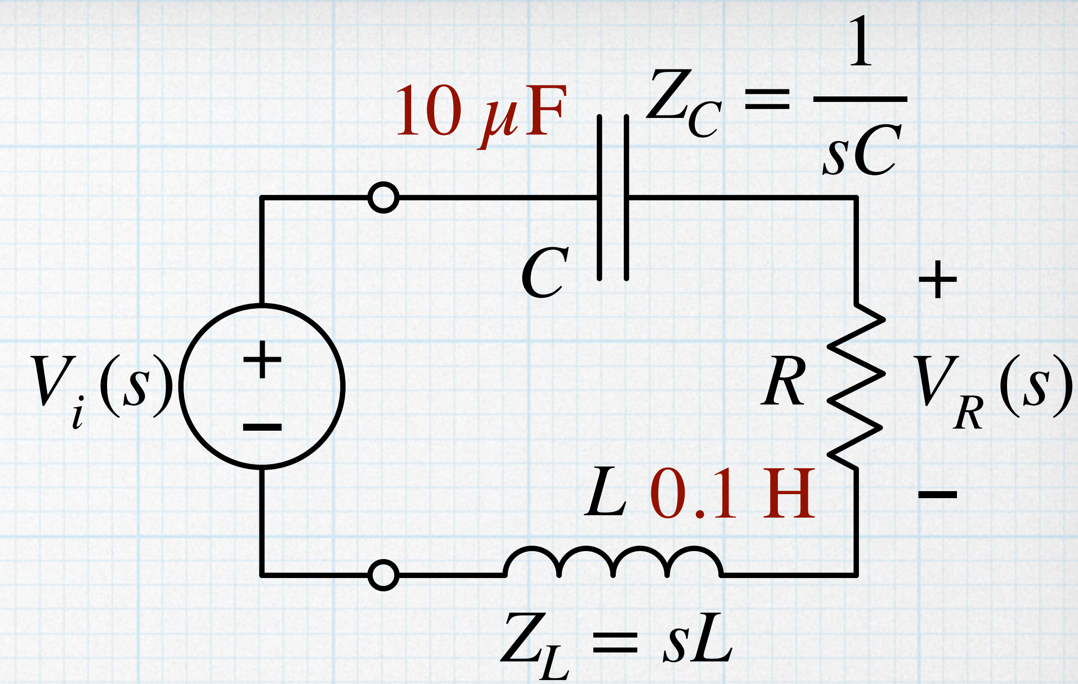
The results are very similar to the simple *RC* divider circuit. That's because the transfer functions are very similar. With the amp, there is some gain and the negative sign introduces an extra 180° of phase, but otherwise the curves are the same.





Example 3

Find the frequency response of the resistor voltage in the *RLC* circuit at right.



This is similar to the *RLC* done previously. In the earlier example, the voltage was taken across the cap.

It is still a voltage divider.

$R = 200\ \Omega$

$$V_R(s) = \left(\frac{R}{\frac{1}{sC} + R + sL} \right) V_i(s)$$

The transfer function is:

$$T(s) = \frac{V_R(s)}{V_i(s)} = \frac{R}{\frac{1}{sC} + R + sL} = \frac{s\left(\frac{R}{L}\right)}{s^2 + s\left(\frac{R}{L}\right) + \frac{1}{LC}}$$

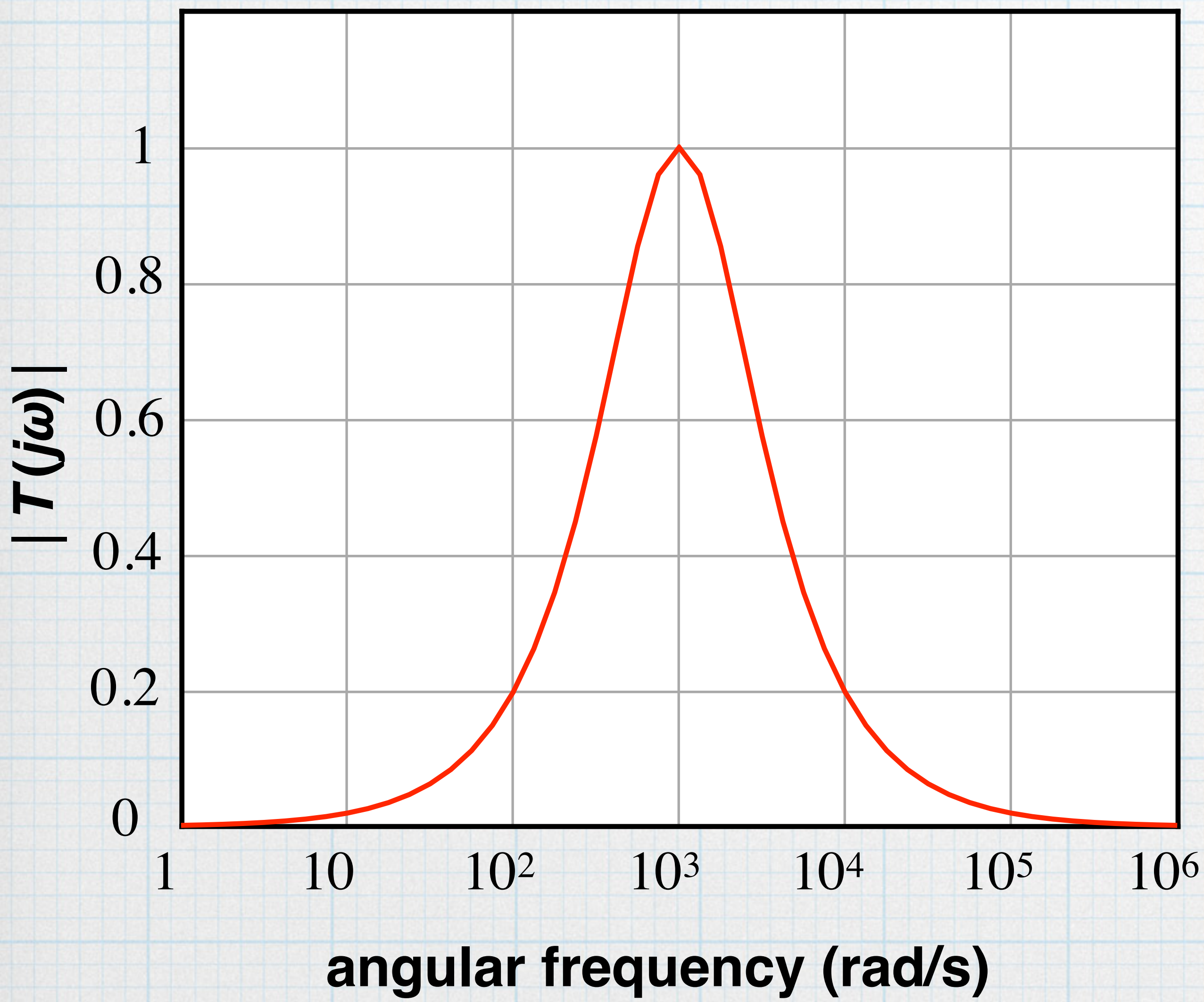
Substitute $s = j\omega$: (Be careful with the s^2 term.)

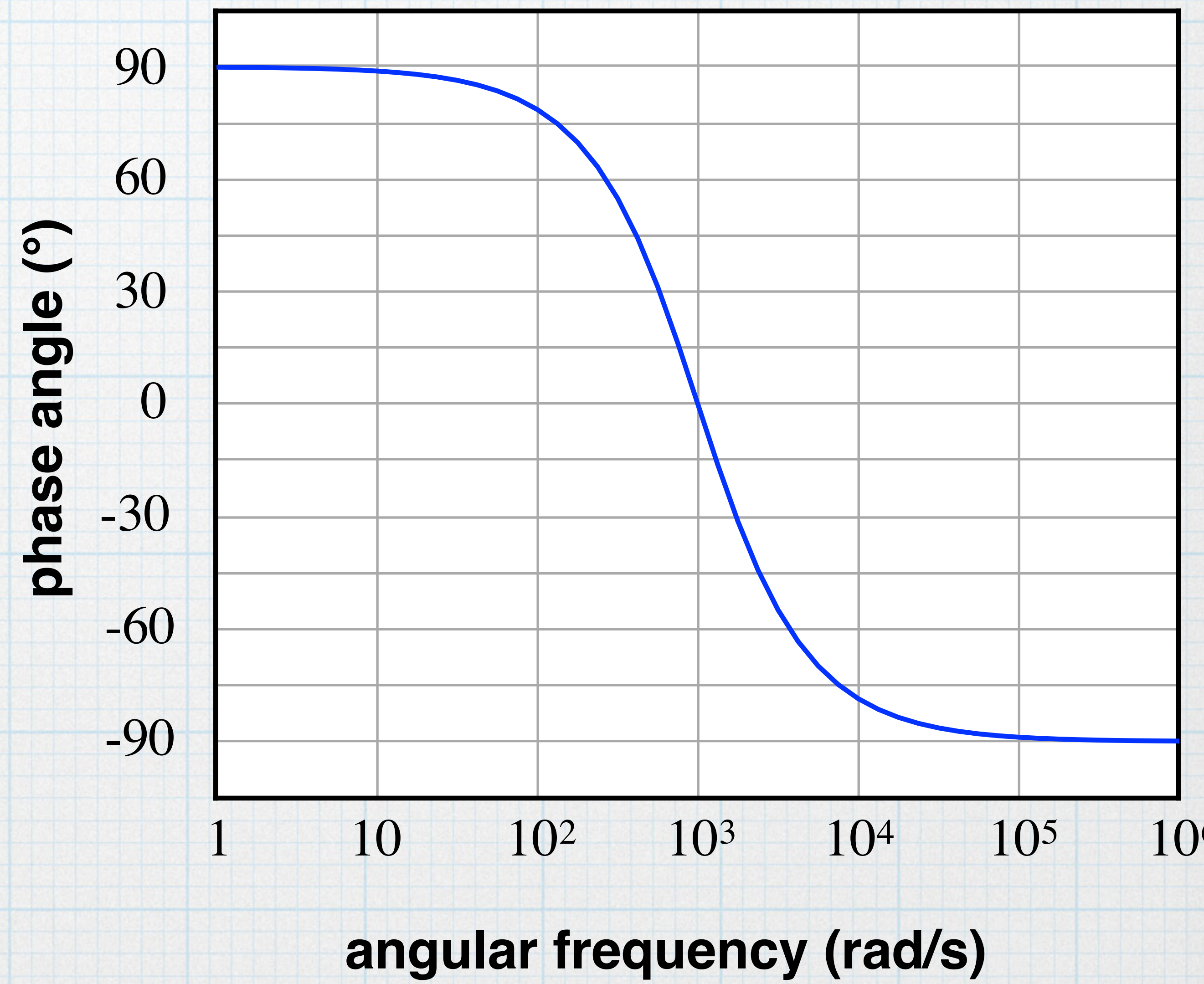
$$T(j\omega) = \frac{j\omega\left(\frac{R}{L}\right)}{-\omega^2 + j\omega\left(\frac{R}{L}\right) + \frac{1}{LC}} = \frac{j\left(\omega\frac{R}{L}\right)}{\left(\frac{1}{LC} - \omega^2\right) + j\left(\omega\frac{R}{L}\right)}$$

Magnitude: $\left|T(j\omega)\right| = \dfrac{\left(\omega \frac{R}{L}\right)}{\sqrt{\left(\frac{1}{LC} - \omega^2\right)^2 + \left(\omega \frac{R}{L}\right)^2}}$ Kinda messy.

Phase: $\theta_T = 90^\circ - \arctan\left(\dfrac{\omega \frac{R}{L}}{\frac{1}{LC} - \omega^2}\right)$

Must be careful to adjust phase properly when $\omega^2 = 1/LC$.





Middle range frequencies have higher amplitude. Low and high frequencies are attenuated. This is band-pass.