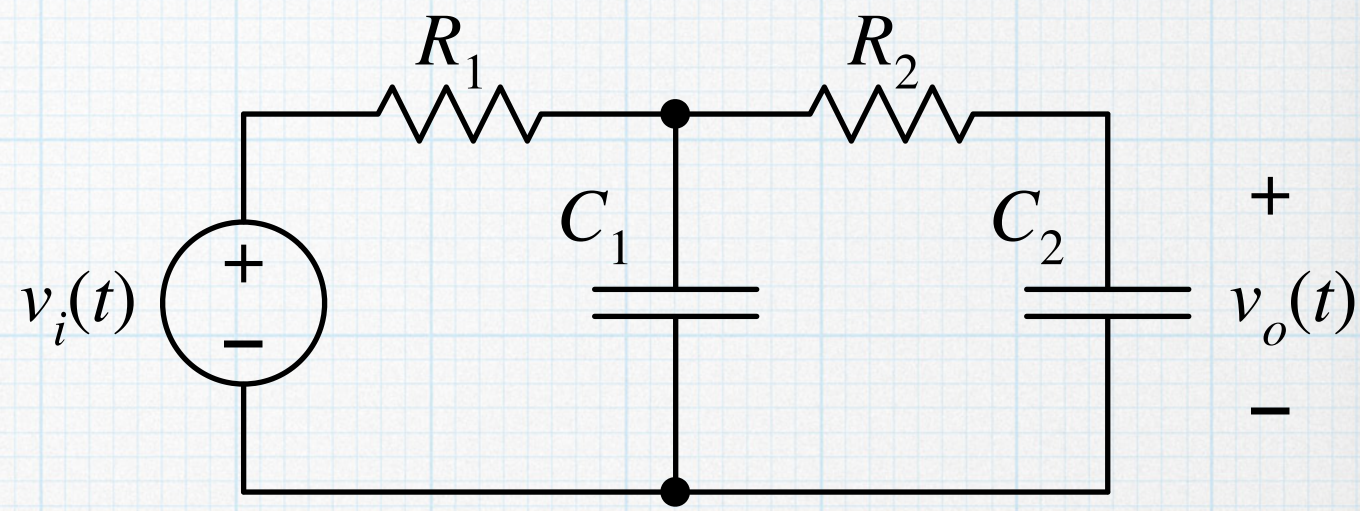
second-order ladder filters — some details

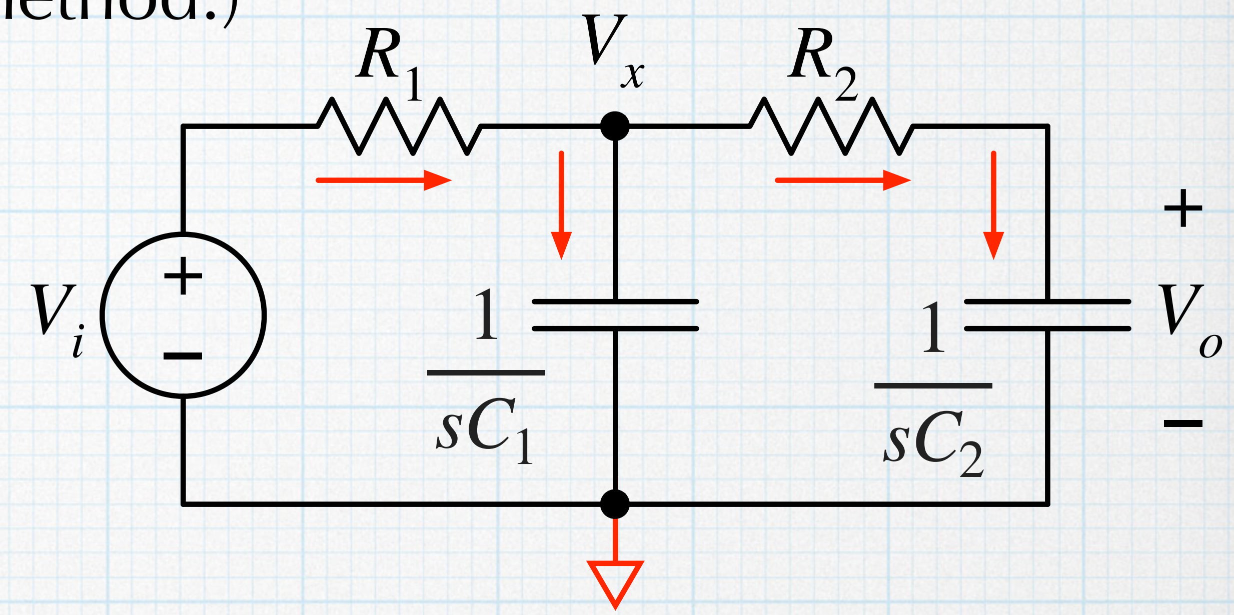
"Ladder"-type filter circuits, like the *RC* circuit shown at right, are quite common and easily implemented. They show up frequently (at least in homework problems and lab exercises) that it is worthwhile to develop a favorite technique for handling them.



We will use three methods — node voltage, mesh current, and "two voltage dividers" — to derive the transfer function. You can then use the method that best resonates with you. (Pardon the pun.) The details of the calculations are specific to this *RC* circuit, but the general approach will apply to ladder circuits with any combination of *R*'s, *L*'s, and *C*'s.

As always, before diving into a pile of math, it is a good idea to take a qualitative look at the circuit. We see that this particular circuit consists two *RC* low-pass sections (capacitors in shunt) connected together, so we expect the second-order response to also be low-pass. Without an inductor, there is no possibility for resonance, and the filter must be limited to $Q_P < 0.5$.

Node voltage (This is my preferred method.)



1. Start with two node-voltage equations, in terms of impedances.

$$\frac{V_i - V_x}{Z_{R1}} = \frac{V_x}{Z_{C1}} + \frac{V_x - V_o}{Z_{R2}}$$

$$\frac{V_x - V_o}{Z_{R2}} = \frac{V_o}{Z_{C2}}$$

2. Insert *R*'s and *C*'s.

$$\frac{V_i - V_x}{R_1} = sC_1V_x + \frac{V_x - V_o}{R_2}$$

$$\frac{V_x - V_o}{R_2} = sC_2V_o$$

3. Re-arrange each equation.

$$V_i = \left[1 + \frac{R_1}{R_2} + sR_1C_1\right] V_x - \frac{R_1}{R_2}V_o$$

$$V_x = \left[1 + sR_2C_2\right] V_o$$

4. Substitute V_x. from the second into the first.

$$V_i = \left[1 + \frac{R_1}{R_2} + sR_1C_1\right]\left[1 + sR_2C_2\right]V_o - \frac{R_1}{R_2}V_o$$

$$V_i = \left[1 + \frac{R_1}{R_2} + sR_1C_1\right]\left[1 + sR_2C_2\right]V_o - \frac{R_1}{R_2}V_o$$

6. Expand the product.

$$V_i = \left[1 + \cancel{\frac{R_1}{R_2}} + sR_1C_1 + sR_2C_2 + sR_1C_2 + s^2R_1R_2C_1C_2\right]V_o - \cancel{\frac{R_1}{R_2}}V_o$$

7. Note that there are two terms that cancel – this always happens. Collect together similar powers and divide as needed to make a transfer function.

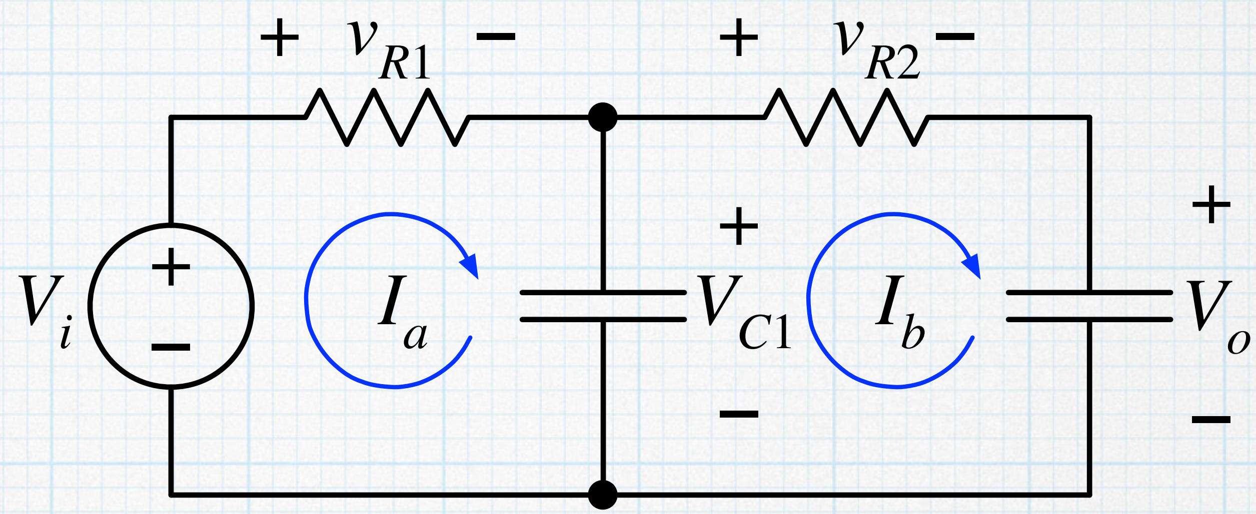
$$T(s) = \frac{V_o}{V_i} = \frac{1}{1 + \left(R_1C_1 + R_2C_2 + R_1C_2\right)s + s^2R_1R_2C_1C_2}$$

8. Finally, re-arrange to put it into standard form:

$$T(s) = \frac{\frac{1}{R_1R_2C_1C_2}}{s^2 + \left(\frac{1}{R_1C_1} + \frac{1}{R_2C_2} + \frac{1}{R_2C_1}\right)s + \frac{1}{R_1R_2C_1C_2}} = \frac{G_o\omega_o^2}{s^2 + \frac{\omega_o}{Q_P}s + \omega_o^2}$$

$$\omega_o^2 = \frac{1}{R_1R_2C_1C_2} \qquad \frac{\omega_o}{Q_P} = \frac{1}{R_1C_1} + \frac{1}{R_2C_2} + \frac{1}{R_2C_1} \qquad G_o = 1$$

Mesh current.



1. Write KVL equations around each loop.

$$V_i - V_{R1} - V_{C1} = 0$$

$$V_{C1} - V_{R2} - V_{C2} = 0$$

$$V_o = I_b Z_{C2} = \frac{I_b}{sC_2}$$

2. Write in terms of the mesh currents and the impedances.

$$V_i - I_a Z_{R1} - \left(I_a - I_b\right) Z_{C1} = 0$$

$$\left(I_a - I_b\right) Z_{C1} - I_b Z_{R2} - I_b Z_{C2} = 0$$

3. Insert *R*'s and *C*'s.

$$V_i - I_a R_1 - \frac{I_a - I_b}{sC_1} = 0$$

$$\frac{I_a - I_b}{sC_1} - I_b R_2 - \frac{I_b}{sC_2} - = 0$$

4. Rearrange.

$$I_a \left(R_1 + \frac{1}{sC_1}\right) - \frac{I_b}{sC_1} = V_i$$

$$-I_a + I_b \left(1 + \frac{C_1}{C_2} + sR_2 C_1\right) = 0$$

5. Eliminate I_a. (Solve second equation of step 4 for I_a and substitute.)

$$I_b\left(1+\frac{C_1}{C_2}+sR_2C_1\right)\left(R_1+\frac{1}{sC_1}\right)I_b-\frac{I_b}{sC_1}=V_i$$

6. Expand the product. Note that two terms cancel.

$$I_b\left(R_1+\frac{R_1C_1}{C_2}+sR_1R_2C_1+\cancel{\frac{1}{sC_1}}+\frac{1}{sC_2}+R_2-\cancel{\frac{1}{sC_1}}\right)=V_i$$

7. Substitute sC_2V_o for I_b.

$$sC_2V_o\left(R_1+\frac{R_1C_1}{C_2}+sR_1R_2C_1+\frac{1}{sC_2}+R_1\right)=V_i$$

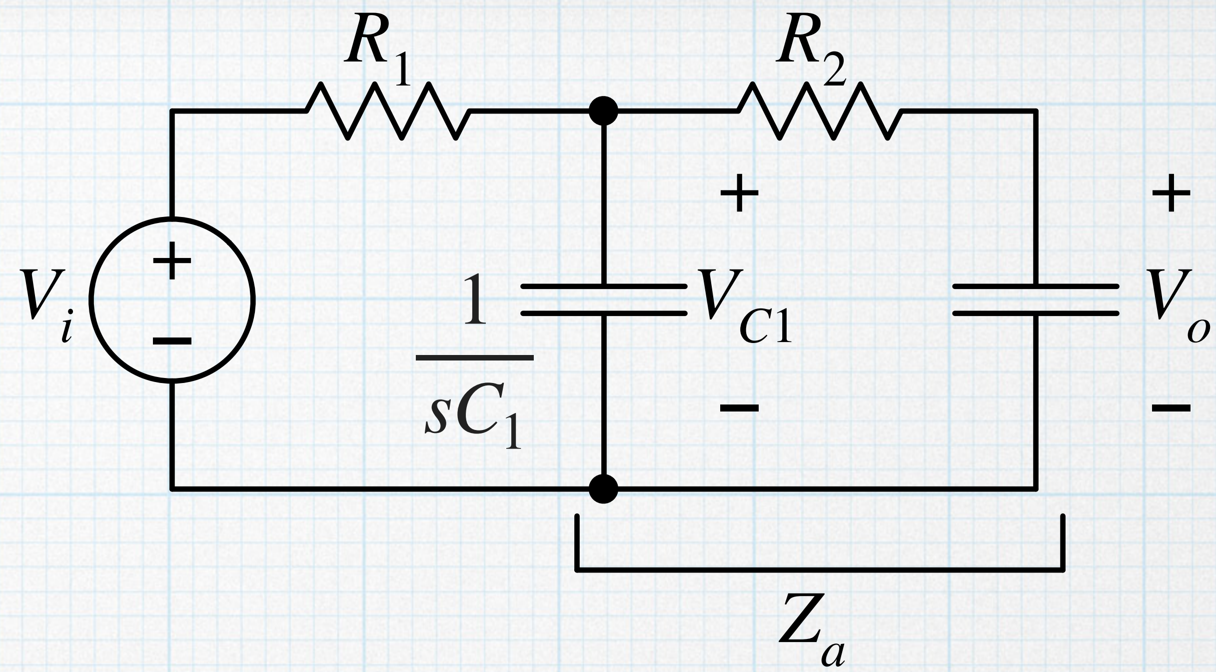
8. Multiply through by sC_2. Re-arrange to form a transfer function

$$T(s)=\frac{V_o}{V_i}=\frac{1}{sC_2R_1+sR_1C_1+s^2R_1R_2C_1C_2+1+sR_1C_2}$$

9. Re-cast into standard form. Same as the node-voltage result.

$$T(s)=\frac{\frac{1}{R_1R_2C_1C_2}}{s^2+\left(\frac{1}{R_1C_1}+\frac{1}{R_2C_2}+\frac{1}{R_2C_1}\right)s+\frac{1}{R_1R_2C_1C_2}}$$

Two voltage dividers

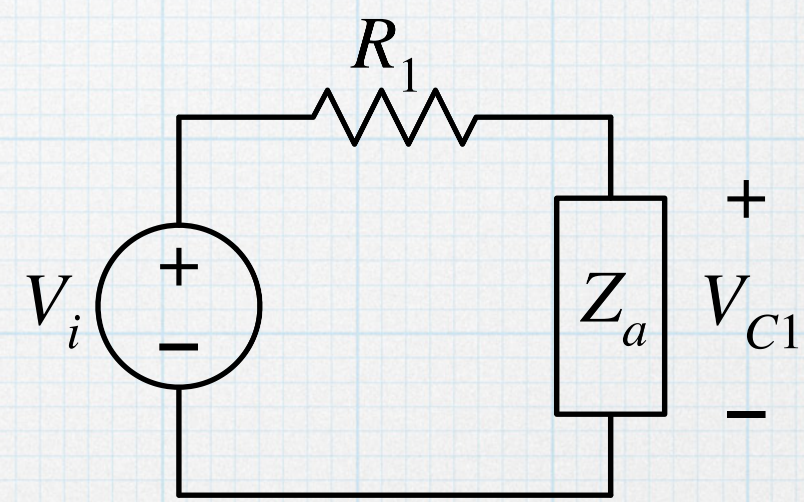


1. Combine Z_{C1}, Z_{R2}, and Z_{C2} in order to find V_{C1}.

$$Z_a = Z_{C1} \parallel (Z_{R2} + Z_{C2})$$

$$= \left(\frac{1}{sC_1}\right) \parallel \left(R_2 + \frac{1}{sC_2}\right)$$

$$= \frac{\frac{1}{sC_1}\left(R_2 + \frac{1}{sC_2}\right)}{\frac{1}{sC_1} + R_2 + \frac{1}{sC_2}}$$



2. Do the first voltage divider to find V_{C1}.

$$V_{C1} = \frac{Z_a}{Z_a + Z_{R1}} V_i = \frac{\frac{\frac{1}{sC_1}\left(R_2 + \frac{1}{sC_2}\right)}{\frac{1}{sC_1} + R_2 + \frac{1}{sC_2}}}{\frac{\frac{1}{sC_1}\left(R_2 + \frac{1}{sC_2}\right)}{\frac{1}{sC_1} + R_2 + \frac{1}{sC_2}} + R_1} V_i$$

3. Simplify!

$$V_{C1} = \frac{\frac{1}{sC_1}\left(R_2 + \frac{1}{sC_2}\right)}{\frac{1}{sC_1}\left(R_2 + \frac{1}{sC_2}\right) + R_1\left(\frac{1}{sC_1} + R_2 + \frac{1}{sC_2}\right)} V_i$$

4. Simplify some more!!

$$V_{C1} = \frac{\left(1 + sR_2C_2\right)}{1 + sR_2C_2 + sR_1C_1 + sR_1C_2 + s^2R_1R_2C_1C_2} V_i$$

5. The second voltage divider gives V_o in terms of V_{C1}.

$$V_o = \frac{Z_{C2}}{Z_{C2} + Z_{R2}} V_{C2} = \frac{\frac{1}{sC_2}}{\frac{1}{sC_2} + R_2} V_{C2} = \frac{1}{1 + sR_2C_2} V_{C2}$$

6. Put the two together. Note the convenient cancellation top and bottom.

$$V_o = \left[\frac{1}{1+sR_2C_2}\right]\left[\frac{\left(1+sR_2C_2\right)}{1+sR_2C_2+sR_1C_1+sR_1C_2+s^2R_1R_2C_1C_2}V_i\right]$$

7. Finally, we have the transfer function:

$$T(s) = \frac{V_o}{V_i} = \frac{1}{1+sR_2C_2+sR_1C_1+sR_1C_2+s^2R_1R_2C_1C_2}$$

8.Put it in standard form.

$$T(s) = \frac{\frac{1}{R_1R_2C_1C_2}}{s^2+\left(\frac{1}{R_1C_1}+\frac{1}{R_2C_2}+\frac{1}{R_2C_1}\right)s+\frac{1}{R_1R_2C_1C_2}}$$

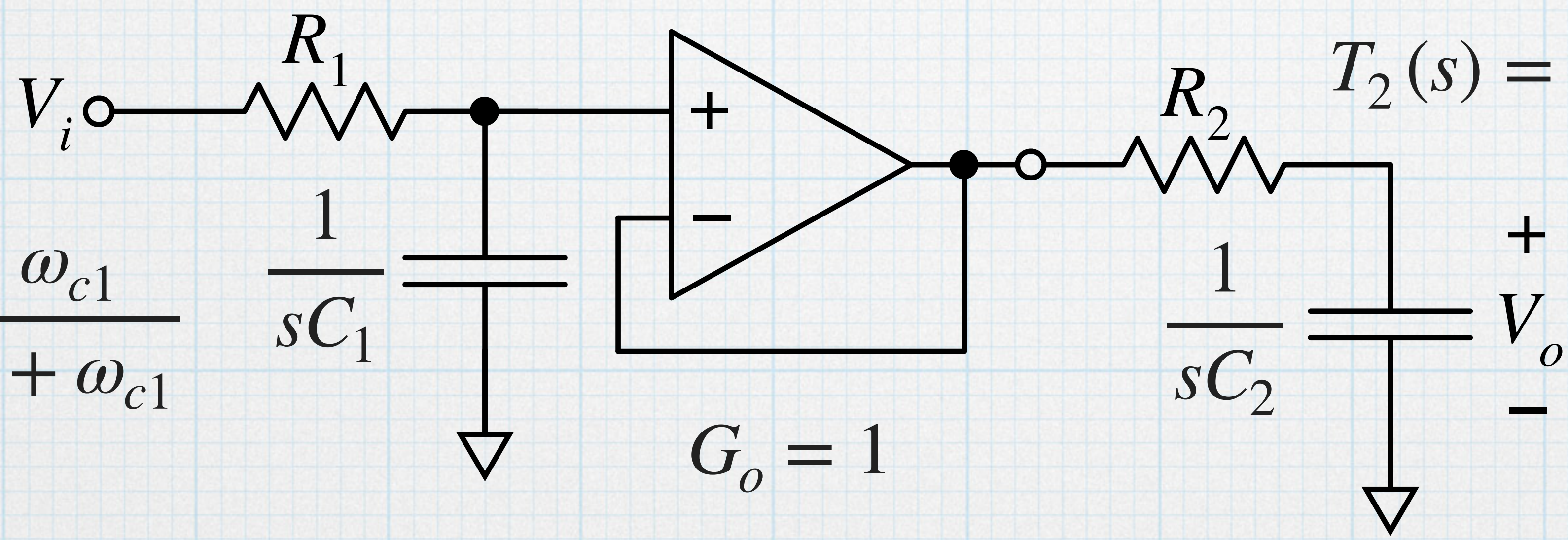
Same.

A buffer makes the math easier

All of the math seems a bit unfair, given that the ladder circuits are essentially combinations of simple first-order circuits. Of course the problem is that the two first-order circuits affect each other once they are connected — they cannot be treated independently. However, if we could keep the two first-order circuits separate in some fashion, then we could simply multiply the first-order functions to obtain the second-order function. Of course, this is exactly what buffer amps were made for. Consider the circuit below.

$$T_2(s) = \frac{\omega_{c2}}{s + \omega_{c2}}$$

$$T_1(s) = \frac{\omega_{c1}}{s + \omega_{c1}}$$



$$T(s) = \left(\frac{\omega_{c1}}{s + \omega_{c1}}\right) \cdot \left(\frac{\omega_{c2}}{s + \omega_{c2}}\right) = \frac{\omega_{c1}\omega_{c2}}{s^2 + \left(\omega_{c1} + \omega_{c2}\right) s + \omega_{c1}\omega_{c2}}$$ Trivial.

Of course, the price for simpler math is the need for an amp and supplies.