Second-order filters

Second-order filters:

- Have second order polynomials in the denominator of the transfer function, and can have zeroth-, first-, or second-order polynomials in the numerator.
- Use two reactive components — 2 capacitors, 2 inductors, or one of each.
- Can be used to make low-pass, high-pass, and *band-pass* frequency responses. (There is also band-reject).
- Have sharper cut-offs than first-order for low-pass and high-pass types. (Clearer distinction between passband and cut-off band.)
- Provide more flexibility in shaping the frequency response.

Second-order filters

Our approach is the same as the first-order circuits.

- Examine the transfer functions for low-pass, high-pass, and (now) band-pass.
- Look at the details of the frequency response of each type of filter — cut-off frequency, pass-band gain, slope in the cut-off band.
- Look at circuits that exhibit the low-pass, high-pass, or band-pass behavior.
- Try some numerical examples to get a feel for the numbers.
- Build and test some circuits in lab.

The general form for the transform function of a second order filter is that of a *biquadratic* (or *biquad* to the cool kids).

$$T(s) = \frac{a_2 s^2 + a_1 s + a_o}{\beta_2 s^2 + \beta_1 s + \beta_o} = G_o \cdot \frac{a_2 s^2 + a_1 s + a_o}{s^2 + b_2 s + b_o} = G_o \cdot \frac{a_2 \left(s + Z_1\right)\left(s + Z_2\right)}{\left(s + P_1\right)\left(s + P_2\right)}$$

As before, G_o is the "gain" of the transfer function. As seen with first-order filters, for passive second-order filters have $G_o \le 1$, and active filters can have gains larger than 1.

The poles of the transfer function determine the general characteristics, and the zeroes determine the filter type. We write the denominator using parameters that will better help us characterize the general behavior.

$$D(s) = s^2 + b_1 s + b_o = s^2 + \left(P_1 + P_2\right) s + P_1 P_2 = s^2 + \left(\frac{\omega_o}{Q_P}\right) s + \omega_o^2$$

where ω_o is the *characteristic frequency*, which determines where things are changing in the frequency response. Important to note: ω_o is not (necessarily) equal to the cut-off frequency. (Details to follow.) Q_P is the *pole quality factor*, and it determines the sharpness of the features in frequency response curve. Note that Q_P has no dimensions.

$$D(s) = s^2 + \left(\frac{\omega_o}{Q_P}\right) s + \omega_o^2$$

Use the quadratic formula to determine the poles.

$$P_{1,2} = -\frac{\omega_o}{2Q_P} \pm \sqrt{\left(\frac{\omega_o}{2Q_P}\right)^2 - \omega_o^2}$$

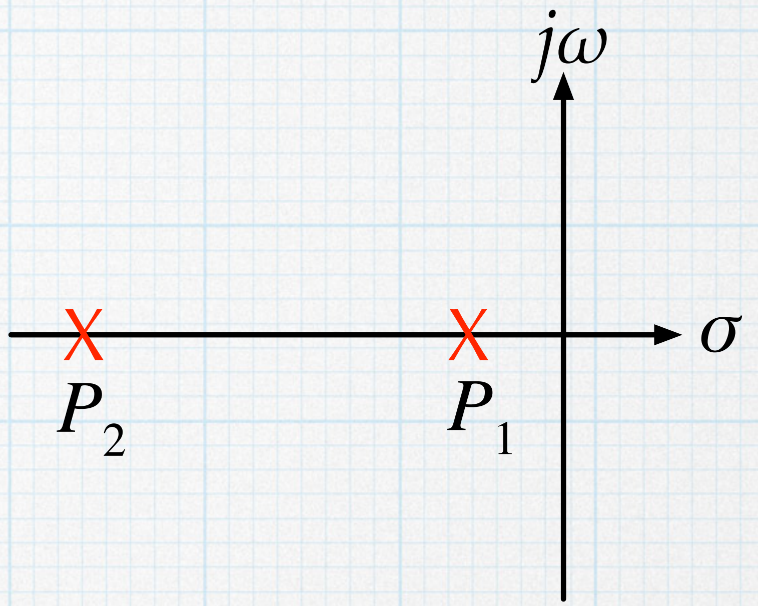
$$= \frac{\omega_o}{2Q_P}\left(-1 \pm \sqrt{1 - 4Q_P^2}\right)$$

The key is the square-root. If the argument under the square-root is positive, there will be two real roots. If the argument is negative, the roots will be a complex conjugate pair. The dividing line is $Q_P = 0.5$:

- $Q_P < 0.5$ → two real, distinct, negative roots.
- $Q_P = 0.5$ → two real, repeated negative roots.
- $Q_P > 0.5$ → complex conjugate roots.

$Q_P < 0.5$

There will be two distinct real roots. The step function response would be an overdamped transient.



$$P_1 = -\frac{\omega_o}{2Q_P} + \sqrt{\left(\frac{\omega_o}{2Q_P}\right)^2 - \omega_o^2}$$

$$P_2 = -\frac{\omega_o}{2Q_P} - \sqrt{\left(\frac{\omega_o}{2Q_P}\right)^2 - \omega_o^2}$$

Example: $P_1 = -200\ \mathrm{s}^{-1}$ and $P_2 = -1800\ \mathrm{s}^{-1}$.

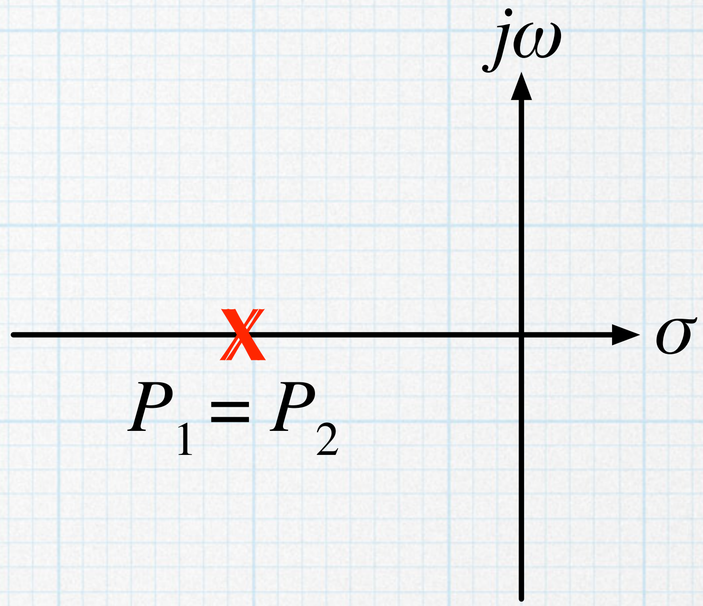
$$D(s) = \left(s + 200\,\mathrm{s}^{-1}\right)\left(s + 1800\,\mathrm{s}^{-1}\right)$$

$$= s^2 + \left(2000\,\mathrm{s}^{-1}\right)s + \left(360{,}000\,\mathrm{s}^{-2}\right)$$

$$D(s) = s^2 + \left(\frac{\omega_o}{Q_P}\right)s + \omega_o^2$$

$$\omega_o = 600\ \mathrm{s}^{-1}\ ;\ \ Q_P = 0.3$$

$Q_P = 0.5$



There are two identical (repeated) real roots. The step function response would be a critically damped transient.

$$P_1 = P_2 = -\frac{\omega_o}{2Q_P}$$

Example: $P_1 = P_2 = -1000\ \mathrm{s}^{-1}$.

$$D(s) = \left(s + 1000\,\mathrm{s}^{-1}\right)^2$$

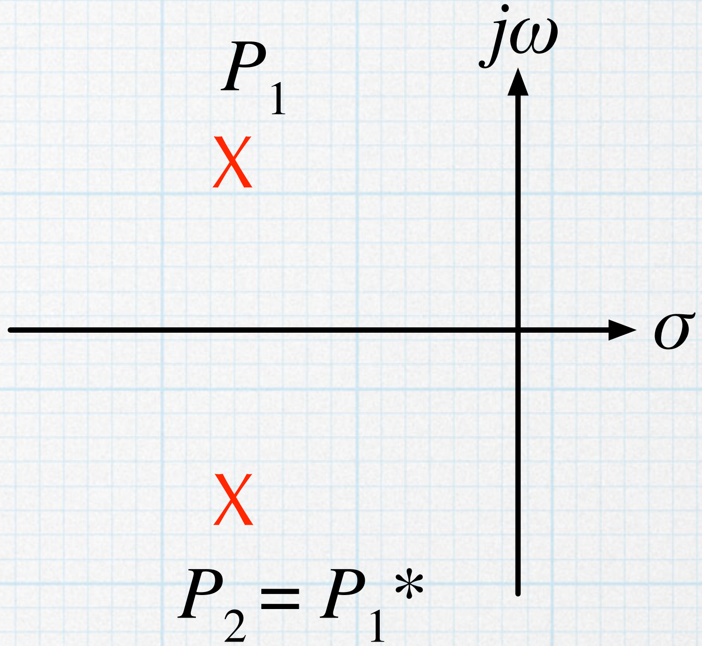
$$= s^2 + \left(2000\,\mathrm{s}^{-1}\right)s + \left(10^6\,\mathrm{s}^{-2}\right)$$

$$D(s) = s^2 + \left(\frac{\omega_o}{Q_P}\right)s + \omega_o^2$$

$$\omega_o = 1000\ \mathrm{s}^{-1}\ ;\ \ Q_P = 0.5$$

$Q_P > 0.5$

The roots will be a complex-conjugate pair. The step response would be an underdamped transient.



$$P_1 = -\frac{\omega_o}{2Q_P} + \sqrt{\omega_o^2 - \left(\frac{\omega_o}{2Q_P}\right)^2}$$

$$P_2 = -\frac{\omega_o}{2Q_P} - \sqrt{\omega_o^2 - \left(\frac{\omega_o}{2Q_P}\right)^2}$$

Example: $P_1 = -1000\ \text{s}^{-1} + j1000\ \text{s}^{-1}$ and $P_2 = -1000\ \text{s}^{-1} - j1000\ \text{s}^{-1}$

$$D(s) = \left(s + 1000\,\text{s}^{-1} + j1000\,\text{s}^{-1}\right)\left(s + 1000\,\text{s}^{-1} - j1000\,\text{s}^{-1}\right)$$

$$= s^2 + \left(2000\,\text{s}^{-1}\right)s + \left(2 \times 10^6\,\text{s}^{-2}\right)$$

$$D(s) = s^2 + \left(\frac{\omega_o}{Q_P}\right)s + \omega_o^2$$

$$\omega_o = 1414\ \text{s}^{-1}\ ;\ \ Q_P = 0.707$$

Low-pass

The form of the numerator determines the type of filter, low-pass, high-pass, band-pass, etc.

$$T(s) = \frac{a_2 s^2 + a_1 s + a_o}{b_2 s^2 + b_1 s + b_o}$$

For a low-pass response, the function should not go to zero as $s \rightarrow 0$, meaning that neither the s or s^2 terms can be in the numerator. Accordingly, we set $a_2 = a_1 = 0$, reducing the numerator the just a constant term. For $s \rightarrow 0$, T is a constant, and $s \rightarrow \infty$, T goes to zero. (Another way of saying this is that the zeros must occur at infinity.) We can arrange the constant from the numerator and use the parameters defined earlier to write the low-pass transfer function as:

$$T_{LP}(s) = G_o \cdot \frac{\omega_o^2}{s^2 + \left(\frac{\omega_o}{Q_P}\right) s + \omega_o^2}$$

As $s \rightarrow 0$: $\frac{\omega_o^2}{s^2 + \left(\frac{\omega_o}{Q_P}\right) s + \omega_o^2} \rightarrow 1$

This may be worth memorizing.

To examine the frequency response, set $s = j\omega$. (Make $G_o = 1$ in order to focus on the rest of the function.)

$$T_{LP}(j\omega) = \frac{\omega_o^2}{-\omega^2 + j\left(\frac{\omega_o}{Q_P}\right)\omega + \omega_o^2} = \frac{\omega_o^2}{\left(\omega_o^2 - \omega^2\right) + j\left(\frac{\omega_o}{Q_P}\right)\omega}$$

Extracting the magnitude and phase.

$$\left|T_{LP}\right| = \frac{\omega_o^2}{\sqrt{\left(\omega_o^2 - \omega^2\right)^2 + \left(\frac{\omega_o}{Q_P} \cdot \omega\right)^2}} \qquad \theta_{LP} = -\arctan\left[\left(\frac{\omega_o}{Q_P}\right)\left(\frac{\omega}{\omega_o^2 - \omega^2}\right)\right]$$

At low frequencies, $|T_{LP}| \approx 1$, as expected. At high frequencies, the magnitude varies inversely with the *square* of the frequency. The phase ranges from 0° at low frequencies to –180° at high frequencies.

Note that for $\omega = \omega_o$, $|T_{LP}| = Q_P$ and $\theta_{LP} = -90°$. Here is an interesting observation: If $Q_p > 1$, then at $\omega = \omega_o$, $|T_{lp}| > 1$!! This requires some further exploration.

The corner frequency is defined in exactly the same manner as with first-order filters,

$$\left|T_{LP}(\omega_c)\right| = 1/\sqrt{2}$$

(Don't forget to include G_o, if $G_o \neq 1$.)

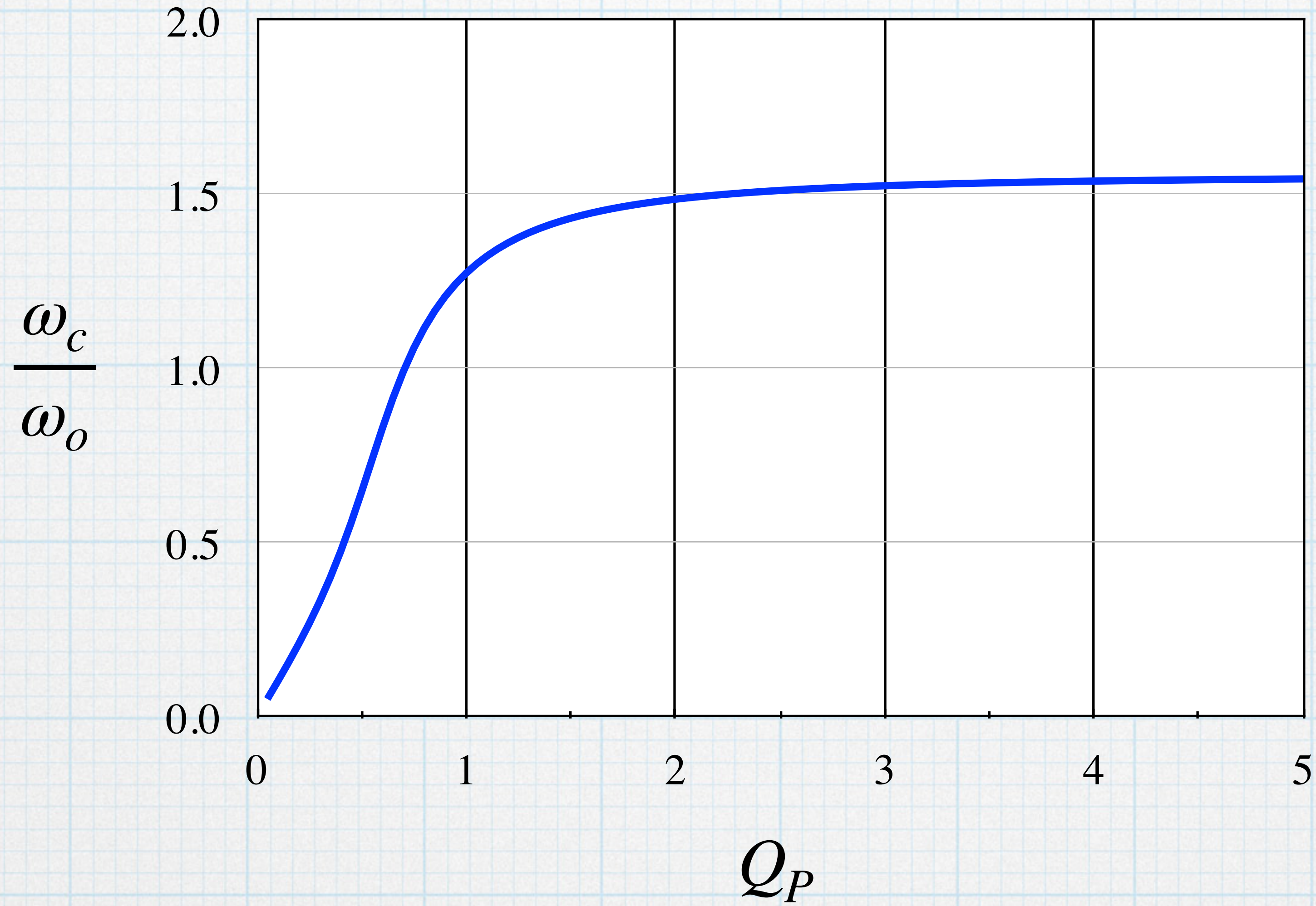
$$\frac{\omega_o^2}{\sqrt{\left(\omega_o^2 - \omega_c^2\right)^2 + \left(\frac{\omega_o}{Q_P} \cdot \omega_c\right)^2}} = \frac{1}{\sqrt{2}}$$

Unfortunately, the math isn't as simple as in the first-order case. After some tedious algebra that includes choosing the correct root when applying the quadratic equation:

$$\omega_c = \omega_o \sqrt{1 - \frac{1}{2Q_P^2} + \sqrt{1 + \left(1 - \frac{1}{2Q_P^2}\right)^2}}$$

Yikes! (Exercise: Derive this for yourself.)

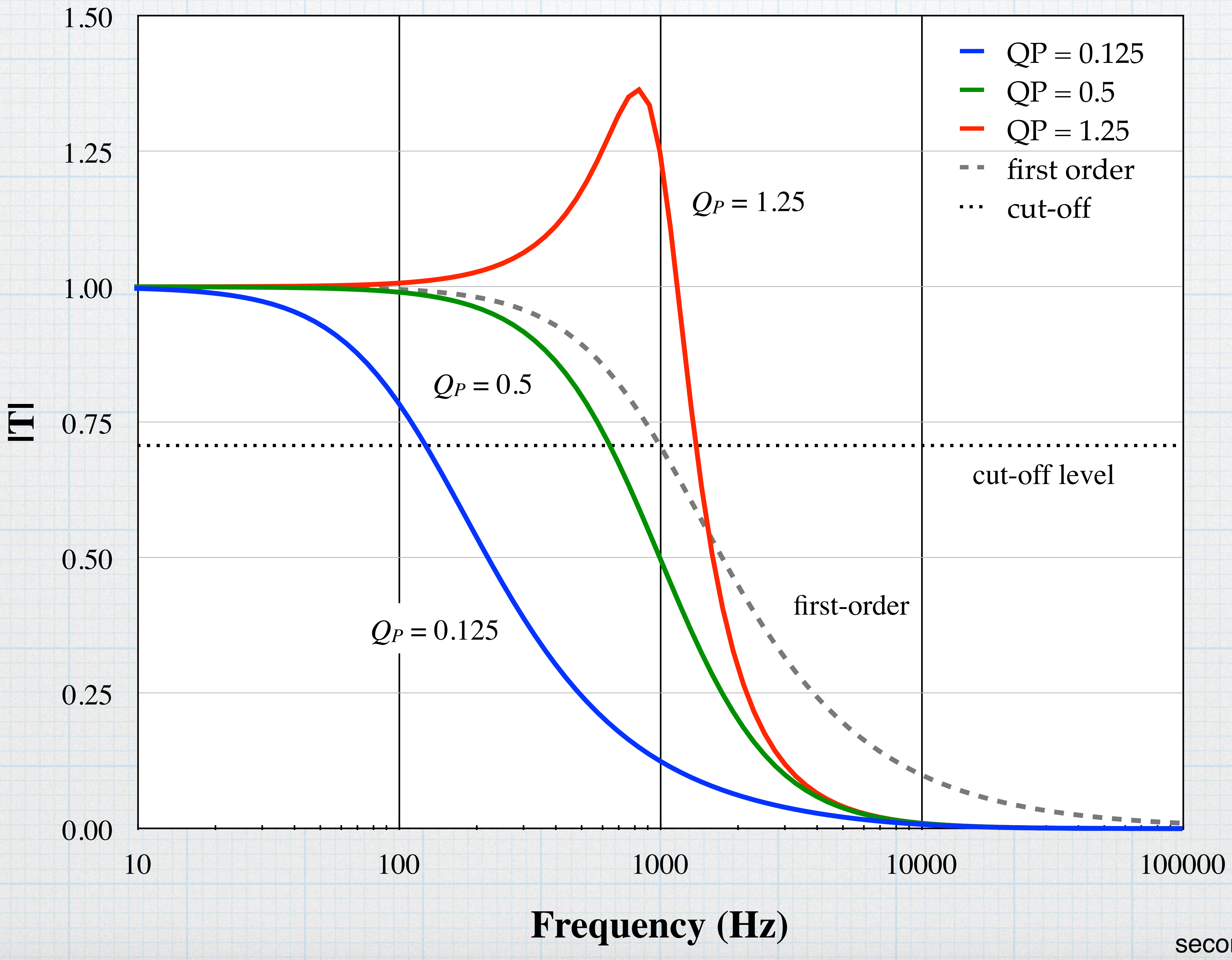
A plot of ω_c / ω_o vs. Q_P. (Expression from the previous slide.)



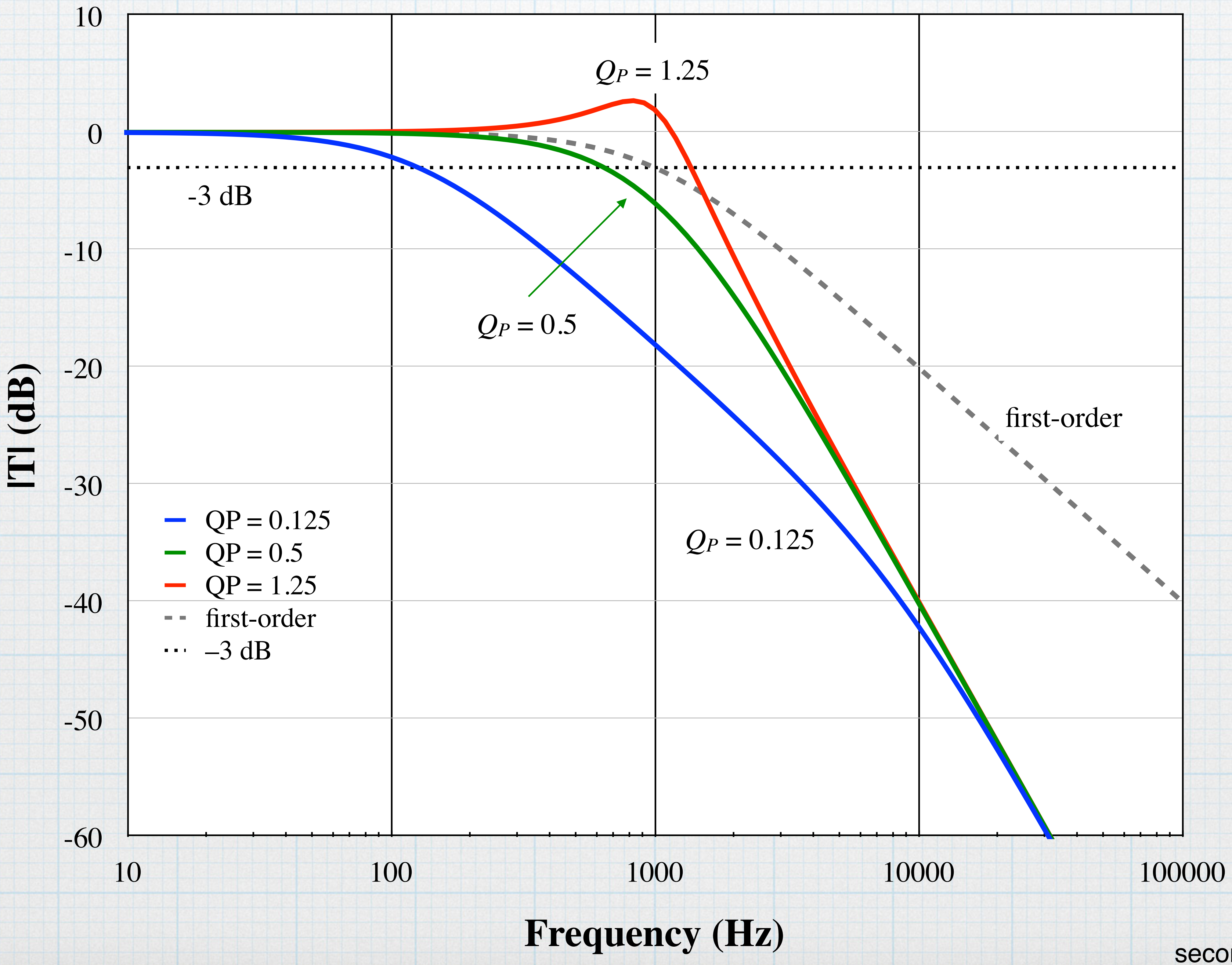
As Q_P become large, $\omega_c = 1.55\omega_o$.

Note that $\omega_c = \omega_o$ when $Q_P = 1/\sqrt{2}$. So in this once instance the corner frequency is easy to find. We will see that this is an important case.

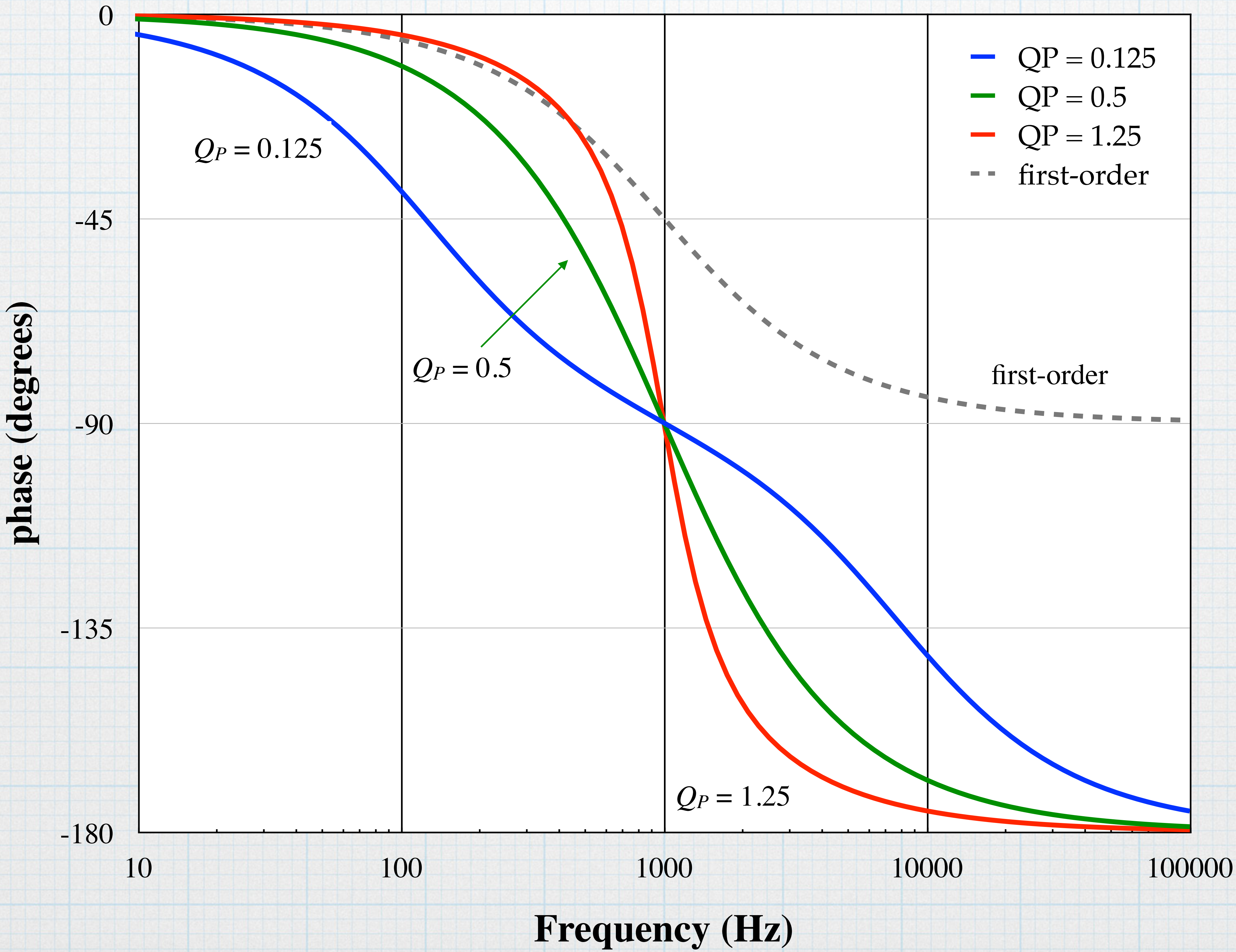
Linear-scale plots of the second-order low-pass magnitude, with $Q_P = 0.125$, 0.5, and 1.25. For all plots, the characteristic frequency is f_o = 1 kHz and the gain is $G_o = 1$. Also shown for comparison is a first-order response with f_c = 1 kHz. What is going on with the magnitude curve for $Q_P = 1.25$?



Bode versions of magnitude plots from previous slide. Note that for frequencies sufficiently high into the cut-off band, the magnitudes decreases at rate of –40 dB/dec — twice the slope for first-order. This is due to $|T| \propto \omega^{-2}$ for $\omega >> \omega_o$.



Phase angle frequency responses for the various cases from the previous slides.



The “bump”

high-pass

$$T(s) = \frac{a_2 s^2 + a_1 s + a_o}{b_2 s^2 + b_1 s + b_o}$$

For a high-pass response we want $T \to 0$ as $s \to 0$ and $T \to$ consant as $s \to \infty$. We accomplish that most easily by setting $a_1 = a_o = 0$ in the numerator. Again, we can manipulate the coefficients and use the parameters defined earlier to write the high-pass function as

$$T_{HP}(s) = G_o \cdot \frac{s^2}{s^2 + \left(\frac{\omega_o}{Q_P}\right) s + \omega_o^2}$$

The polynomial ratio varies from 0 to 1 as s increases from 0 to ∞. The gain, G_o, takes care of the constant value at high frequencies, as determined by any amplifiers or voltage dividers within the circuit.

This also may be worth memorizing.

To examine the frequency response, set $s = j\omega$. (And again, setting $G_o = 1$.)

$$T_{HP}(j\omega) = \frac{-\omega^2}{-\omega^2 + j\left(\frac{\omega_o}{Q_P}\right)\omega + \omega_o^2} = \frac{-\omega^2}{\left(\omega_o^2 - \omega^2\right) + j\left(\frac{\omega_o}{Q_P}\right)\omega}$$

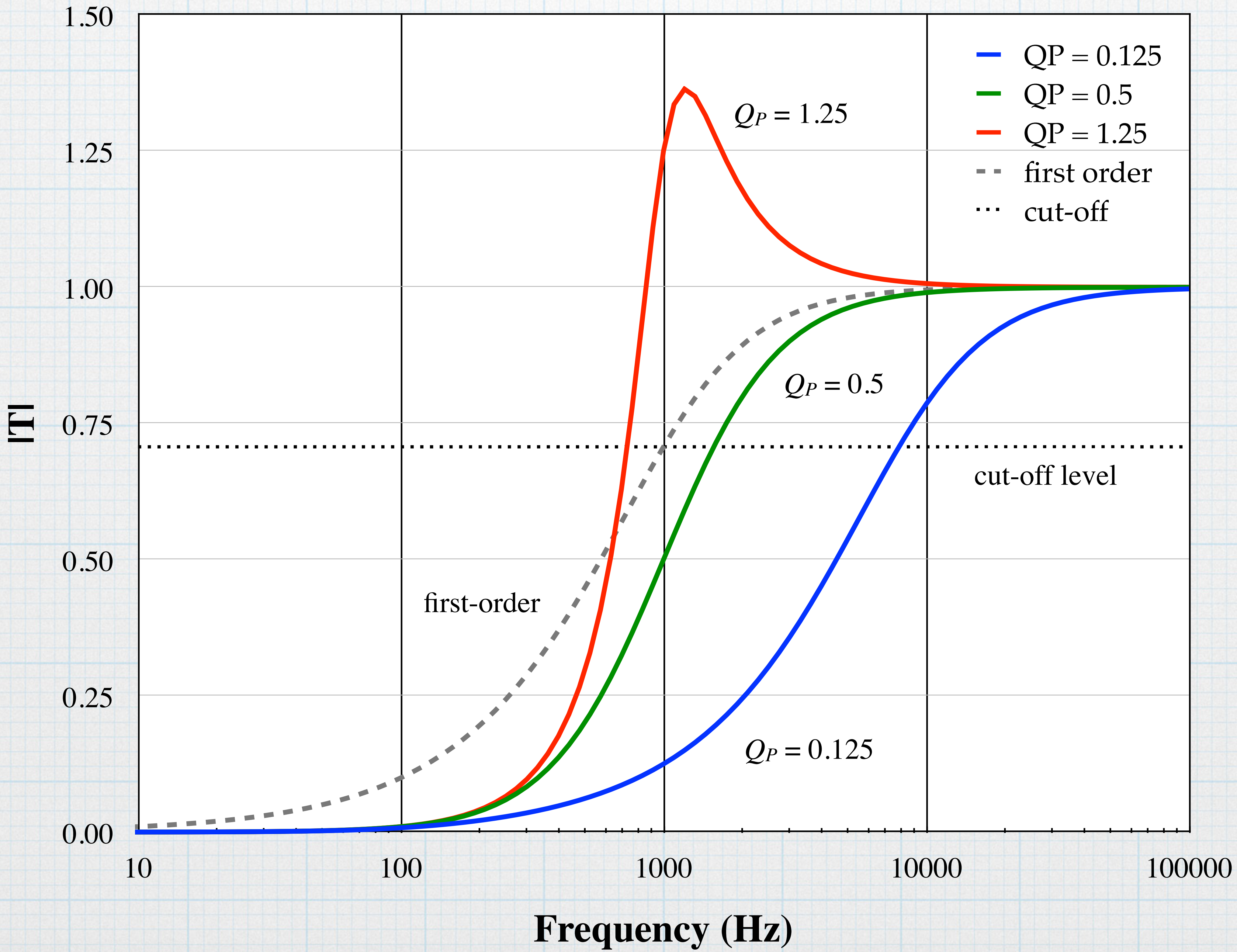
Extracting the magnitude and phase.

$$\left|T_{HP}\right| = \frac{\omega^2}{\sqrt{\left(\omega_o^2 - \omega^2\right)^2 + \left(\frac{\omega_o}{Q_P} \cdot \omega\right)^2}} \qquad \theta_{HP} = 180^\circ - \arctan\left[\left(\frac{\omega_o}{Q_P}\right)\left(\frac{\omega}{\omega_o^2 - \omega^2}\right)\right]$$

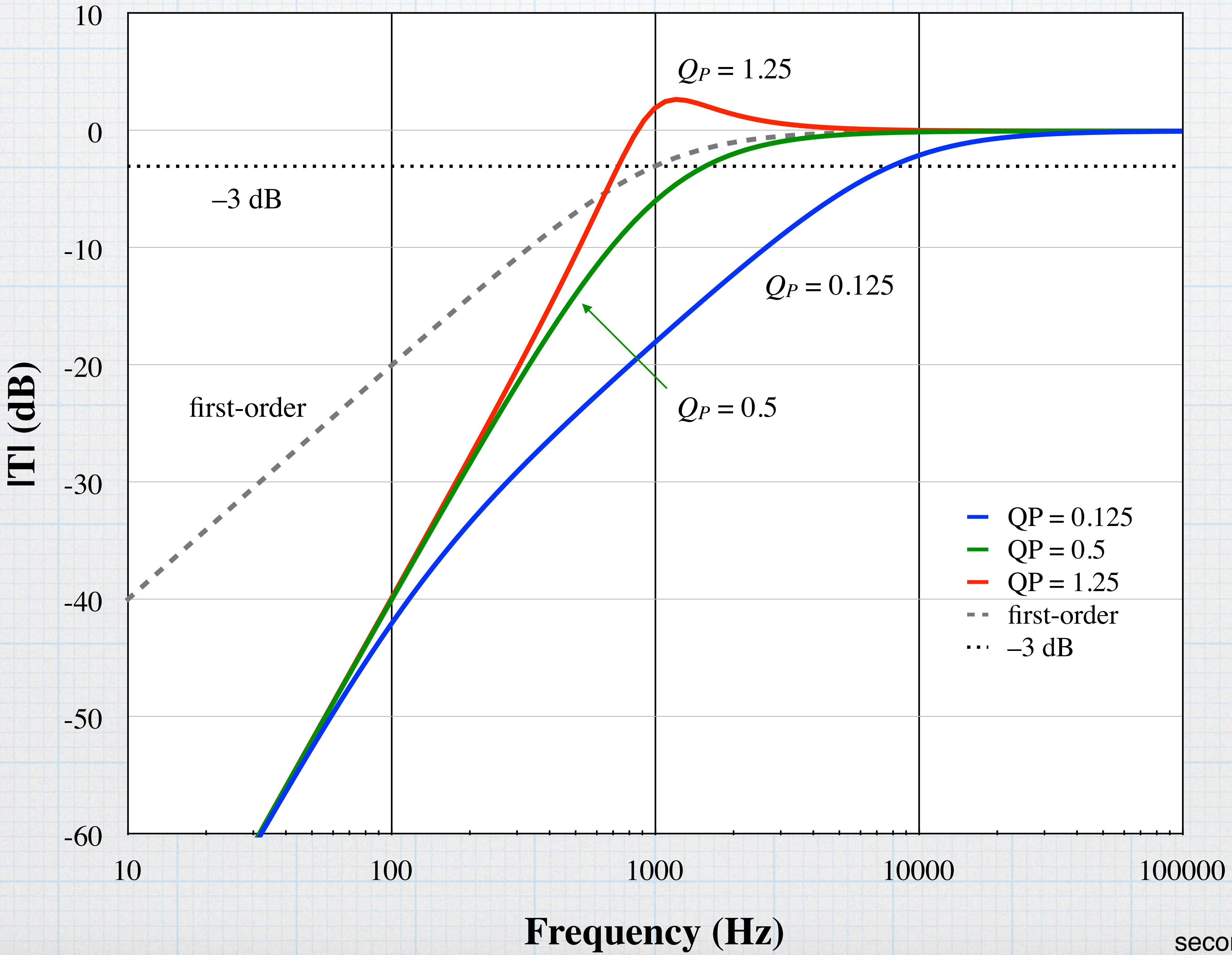
The 180° comes from the negative sign in the numerator of the T.F.

These expressions are very similar to the corresponding low-pass functions. The resulting plots are mirror images of the low-pass plots. The expressions for the cut-off frequency and the properties of the "bump" in the high-Q_P plots are similarly symmetric to those from the low-pass case.

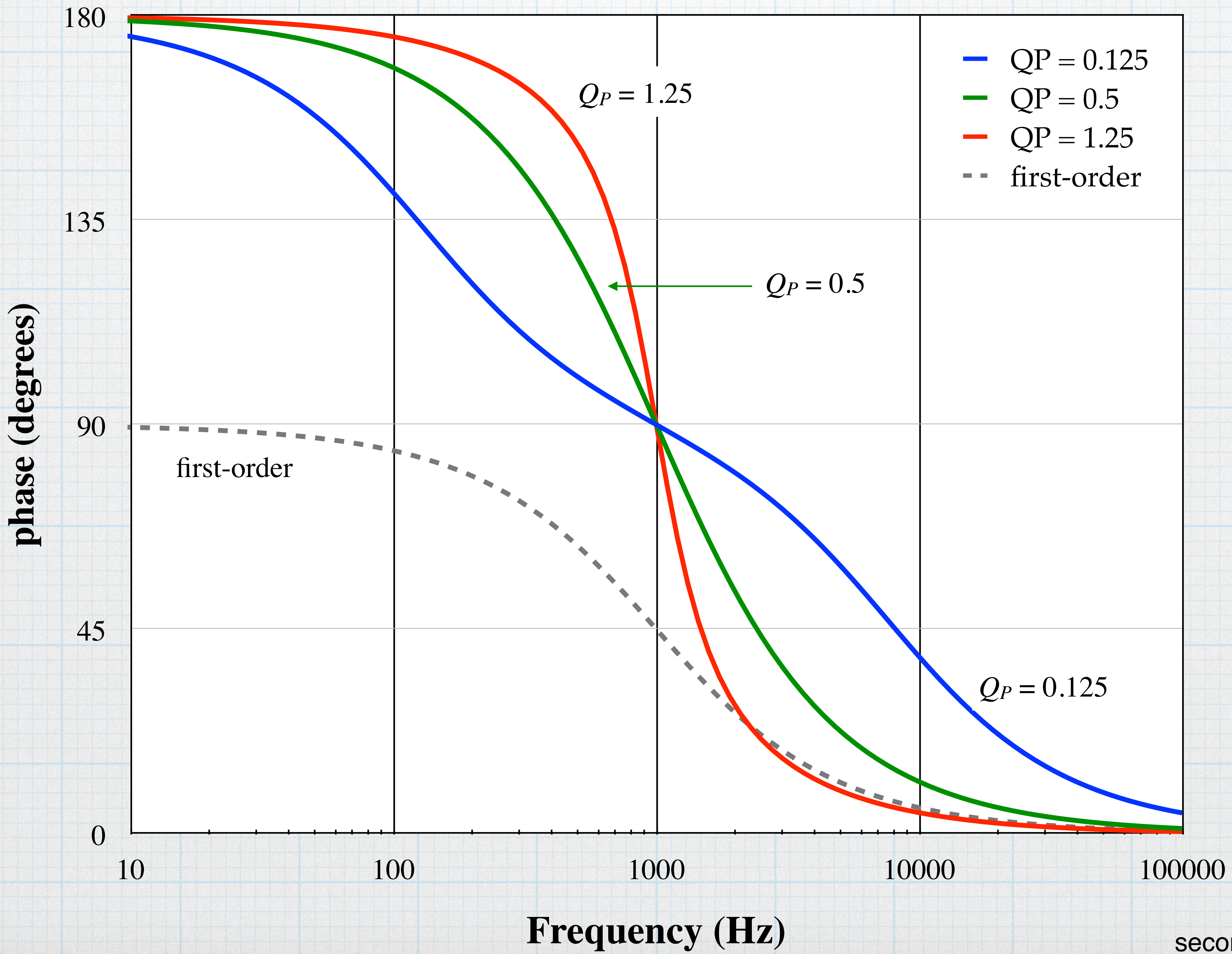
Linear-scale plots of the second-order high-pass magnitude, with $Q_P = 0.125, 0.5,$ and 1.25. For all plots, the characteristic frequency is f_o = 1 kHz and the gain is G_o = 1. Also shown for comparison is a first-order response with f_c = 1 kHz.



Bode versions of magnitude plots from previous slide. As seen with the high-pass Bode plot, there is a slope of 40 dB/dec in the cut-off region, indicative of an inverse-squared relationship.



Phase angle frequency responses for the various cases of the second-order high-pass functions.



band-pass

$$T(s) = \frac{a_2 s^2 + a_1 s + a_o}{b_2 s^2 + b_1 s + b_o}$$

For a band-pass response, the function should go to zero both for $s = 0$ and $s \rightarrow \infty$. This can be accomplished by setting $a_2 = a_o = 0$ in the numerator, keeping only the linear term. Once again, we can combine the coefficients and use the standard 2nd-order form come up with the band-pass transfer function.

$$T_{BP}(s) = G_o \cdot \frac{\left(\frac{\omega_o}{Q_P}\right) s}{s^2 + \left(\frac{\omega_o}{Q_P}\right) s + \omega_o^2}$$

This too may be worth memorizing. (But if you note the symmetry of the various expressions, remembering them all is easy.)

To examine the frequency response, set $s = j\omega$. (Letting $G_o = 1$, again.)

$$T_{BP}(j\omega) = \frac{j\left(\frac{\omega_o}{Q_P}\right)\omega}{-\omega^2 + j\left(\frac{\omega_o}{Q_P}\right)\omega + \omega_o^2} = \frac{j\left(\frac{\omega_o}{Q_P}\right)\omega}{\left(\omega_o^2 - \omega^2\right) + j\left(\frac{\omega_o}{Q_P}\right)\omega}$$

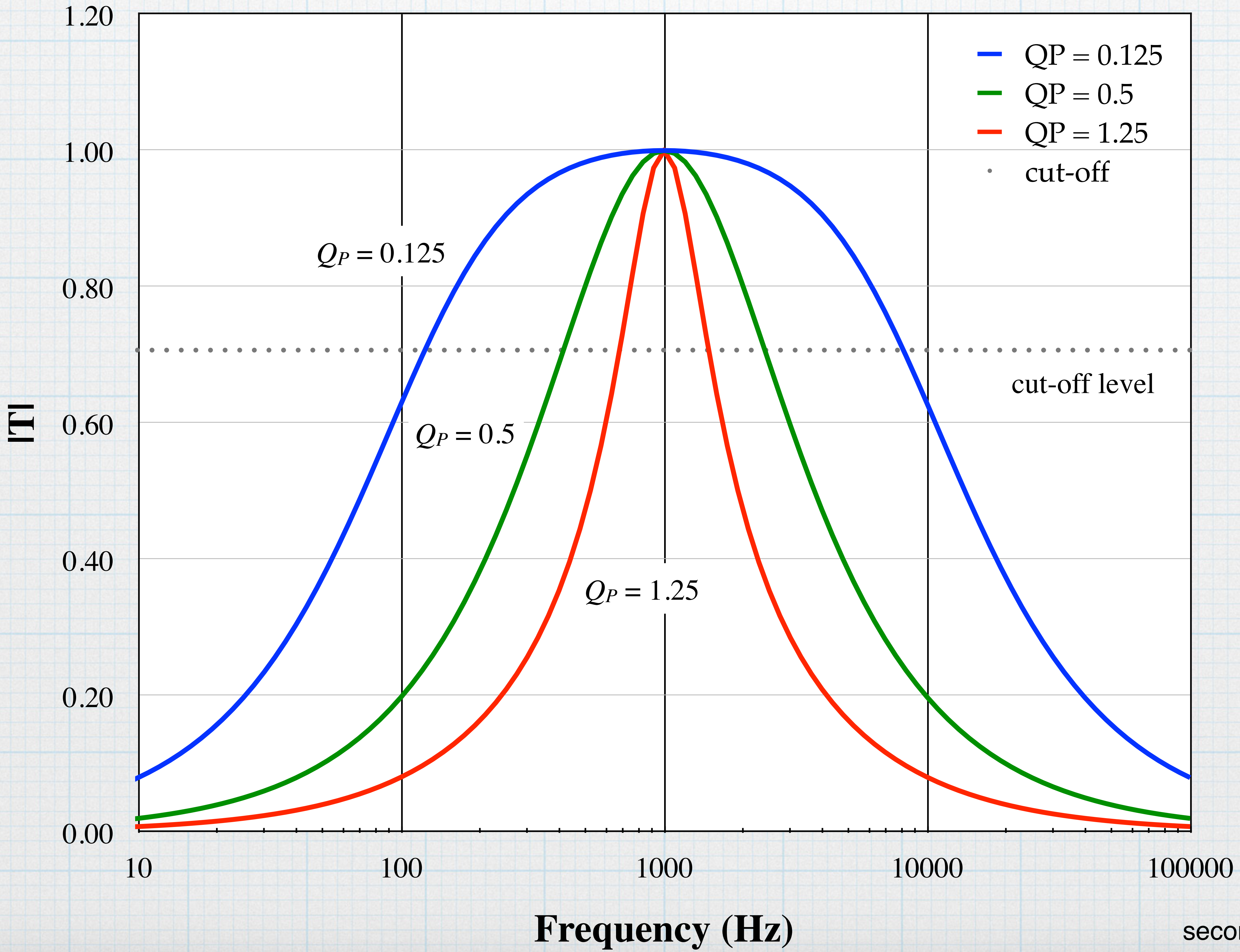
Extracting the magnitude and phase.

$$\left|T_{BP}\right| = \frac{\left(\frac{\omega_o}{Q_P}\right)\omega}{\sqrt{\left(\omega_o^2 - \omega^2\right)^2 + \left(\frac{\omega_o}{Q_P}\cdot\omega\right)^2}} \qquad \theta_{BP} = 90^\circ - \arctan\left[\left(\frac{\omega_o}{Q_P}\right)\left(\frac{\omega}{\omega_o^2 - \omega^2}\right)\right]$$

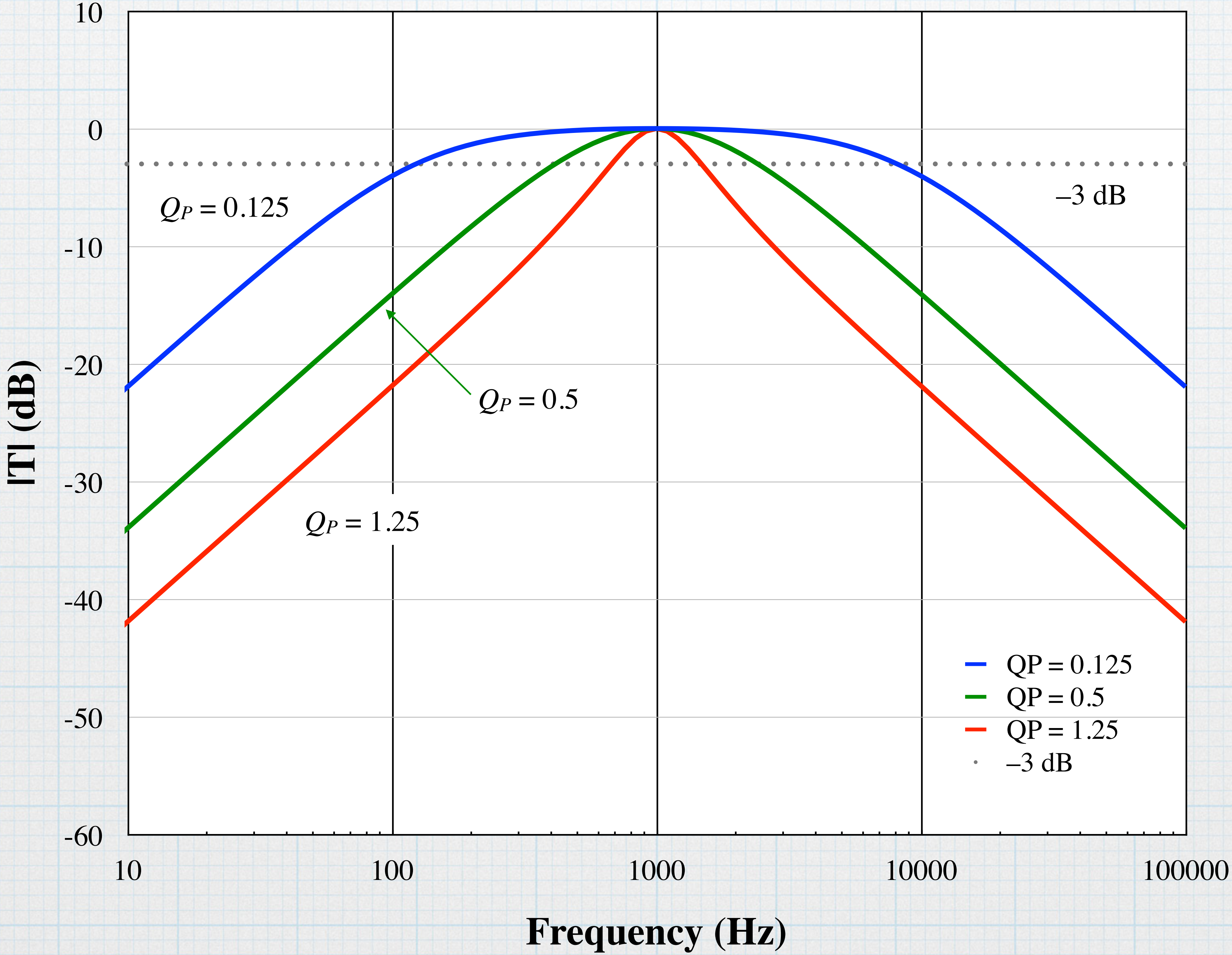
The 90° comes from the imaginary value in the numerator of the T.F.

Again, very similar expressions to the low-pass and high-pass cases.

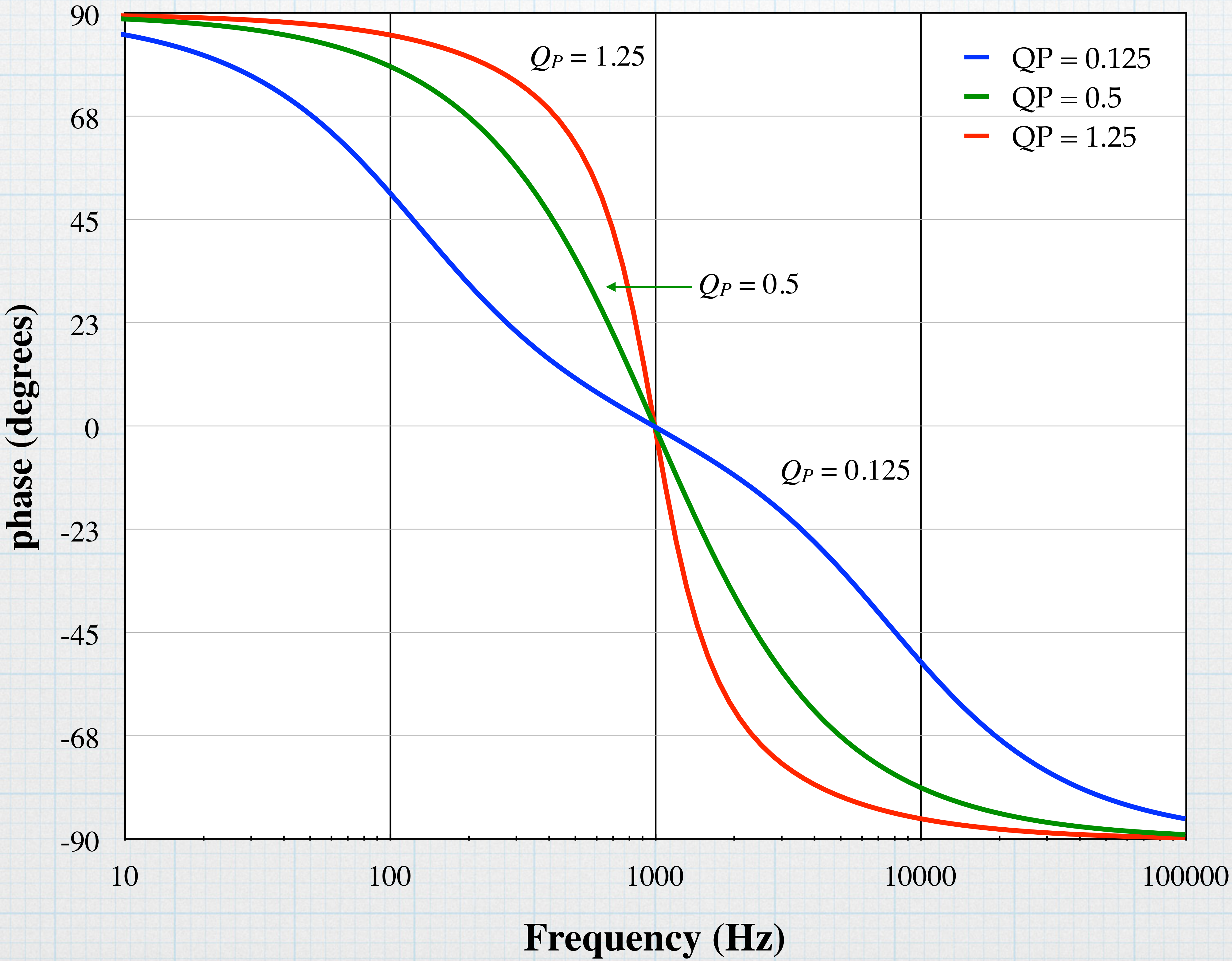
Linear-scale plots of the second-order band-pass magnitude, with $Q_P = 0.125$, 0.5, and 1.25. For all plots, the characteristic frequency is $f_o = 1$ kHz and the gain is $G_o = 1$. (There is no first-order curve for comparison — first-order BP filters do not exist.)



Bode versions of magnitude plots from previous slide. In the high- and low-frequency cut-off bands, the slope is ±20 dB/dec.



Phase angle frequency responses for the various cases of the second-order high-pass functions.



The peak of the bandpass magnitude occurs at the characteristic frequency and falls off at both higher and lower frequencies. Thus there are two corner frequencies, defined in the usual manner. The difference between the two frequencies is the *bandwidth* of the filter. To calculate the two corners and the bandwidth, we start in the usual manner:

$$\frac{\left(\frac{\omega_o}{Q_P}\right)\omega_c}{\sqrt{\left(\omega_o^2-\omega_c^2\right)^2+\left(\frac{\omega_o}{Q_P}\cdot\omega_c\right)^2}}=\frac{1}{\sqrt{2}}$$

After a fair amount of tedious — and sometimes tricky — algebra, we find the high and low corner frequencies, which we denote as ω_{c+} and ω_{c-}.

$$\omega_{c+}=\omega_o\left(\sqrt{1+\frac{1}{4Q_P^2}}+\frac{1}{2Q_P}\right) \qquad \omega_{c-}=\omega_o\left(\sqrt{1+\frac{1}{4Q_P^2}}-\frac{1}{2Q_P}\right)$$

The bandwidth (BW) is the difference between these corners. It turns out to have a surprisingly simple relationship to ω_o and Q_P:

bandwidth (BW): $\Delta\omega=\omega_{c+}-\omega_{c-}=\dfrac{\omega_o}{Q_P}$

The quality factor directly controls the bandwidth.