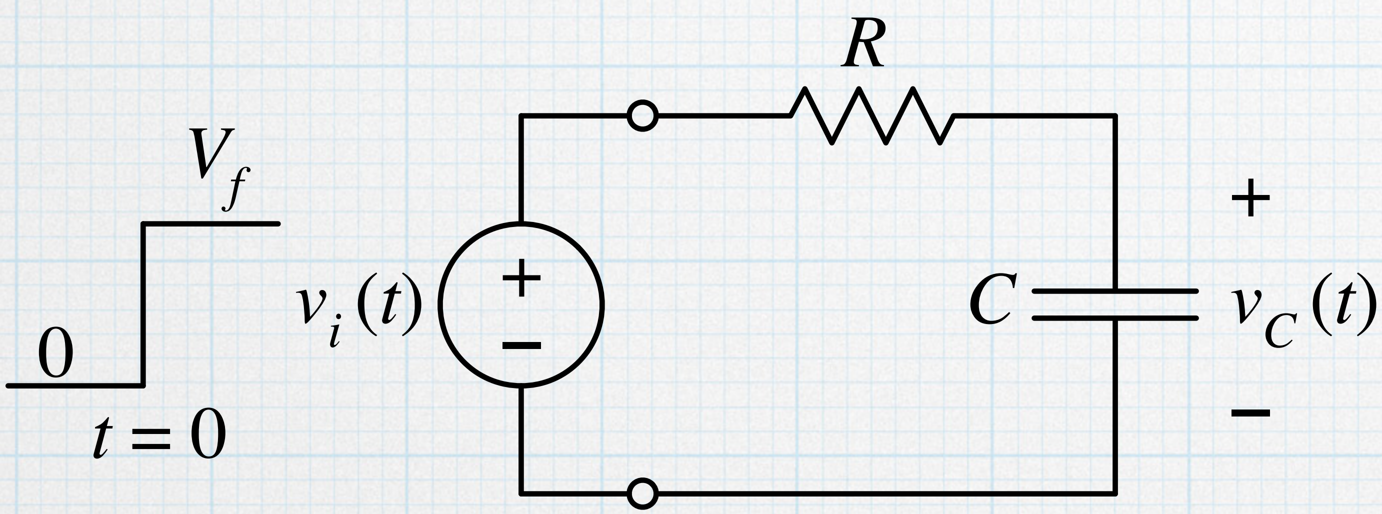
Inverse transform – back to the time domain

In the circuit examples of the previous set of notes, we saw the ease of analyzing circuits in the frequency domain. One example was the RC with a step-voltage source.



$$V_C(s) = \left(\frac{\frac{1}{RC}}{s+\frac{1}{RC}}\right)\left(\frac{V_f}{s}\right)$$

$$= \frac{V_f/_{RC}}{s\left(s+\frac{1}{RC}\right)}$$

Once we obtained the frequency-domain result, then what? Obviously, we need to go in reverse and transform back the time domain.

However, we will state once again that transforming back to the time is not always a necessity. Often, the frequency-domain expression gives us important information about the circuit's behavior.

There is a rigorous method for going back to the time domain. The *inverse Laplace transform* has the form:

$$v_C(t) = \mathscr{L}^{-1}\left\{V_C(s)\right\} = \frac{1}{j2\pi}\int_{\sigma-j\infty}^{\sigma+j\infty} V_C(s)\, e^{st} ds$$

Yikes! Unfortunately, we have entered the realm of *contour integration*. Recall that s is a complex variable, $s = \sigma + j\omega$, and so $V_C(s)$ is complex function of a complex variable. To evaluate the integral, we need to define a contour or path through the complex plane and evaluate the integral along that path. In this case the particular path would a vertical line extending from $-j\infty$ to $+j\infty$, with σ chosen to guarantee that the integral converges. (What does this even mean?) Unless we learn more about complex analysis (Math 365) and the techniques for doing complex integration, we are unable to utilize the above transform.

If contour integration seems too messy, then what? Fortunately, there is a more practical approach. We have a small table of transform pairs built up by finding the Laplace transforms of a few common time-domain functions. We can use the table "backwards". Given a frequency-domain function, we find its entry in the table. The corresponding time-domain function is the inverse transform.

The frequency-domain solution to the *RC* circuit example is

$$V_C(s) = \frac{V_f}{s(1+sRC)} = \frac{V_f}{RC} \cdot \frac{1}{s\left(s+\frac{1}{RC}\right)}$$

Unfortunately, there is nothing in the table that looks like $\dfrac{1}{s(s+a)}$.

(For reference, the table is reproduced on the next slide.)

We have two options. We could expand our table to have more entries, including the one above. Or we could make the above function look more like entries already in the table. We will try the latter.

Some transforms

	$f(t)$	$F(s)$
impulse	$\delta(t)$	1
step	$u(t)$	$\frac{1}{s}$
ramp	t	$\frac{1}{s^2}$
exponential	$e^{-\sigma t}$	$\frac{1}{s+\sigma}$
sine	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
cosine	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
damped ramp	$t \cdot e^{-\sigma t}$	$\frac{1}{(s+\sigma)^2}$
damped sine	$e^{-\sigma t} \sin \omega t$	$\frac{\omega}{(s+\sigma)^2+\omega^2}$
damped cosine	$e^{-\sigma t} \cos \omega t$	$\frac{(s+\sigma)}{(s+\sigma)^2+\omega^2}$

Partial fractions

The advantage of solving circuits in the frequency domain is that the algebra leads to functions that are ratios of polynomials.

$$F(s) = \frac{N(s)}{D(s)} = \frac{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_o}{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_o}$$

In principle, polynomials can always be re-written as product of individual factors. The number of factors of the polynomial will be equal to the highest order of the polynomial. The process of factoring might be messy, the general idea always holds.

$$F(s) = \frac{(s + Z_n)(s + Z_{n-1})\ldots(s + Z_1)(s + Z_o)}{(s + P_m)(s + P_{m-1})\ldots(s + P_1)(s + P_o)}$$

In the above ratio, the various value of Z_i are the *zeroes* of the function — if $s = -Z_i$, then $F = 0$. The various values of P_i are called the *poles* of the function and are the values where the denominator goes to zero. When that happens the function goes to infinity — if $s = -P_i$, $F \rightarrow \infty$.

$$F(s) = \frac{(s+Z_n)(s+Z_{n-1})\ldots(s+Z_1)(s+Z_o)}{(s+P_m)(s+P_{m-1})\ldots(s+P_1)(s+P_o)}$$

The key observation is that a ratio of polynomials can always be written as a *sum of partial fractions:*

$$F(s) = \frac{A_m}{(s+P_m)} + \frac{A_{m-1}}{(s+P_{m-1})} + \ldots + \frac{A_1}{(s+P_1)} + \frac{A_o}{(s+P_o)}$$

Where the A_i coefficients are yet to be determined. All of the terms are very similar, and the form is one that shows up in our little table of transforms! Transforming back to the time-domain, term-by-term

$$f(t) = A_m \exp(-P_m t) + A_{m-1} \exp(-P_{m-1} t) + \ldots + A_1 \exp(-P_1 t) + A_o \exp(-P_o t)$$

The result is a sum of exponential terms. The basic form of the time-domain expression is determined by the poles of the frequency-domain function. The denominator of F(s) determines the time-domain behavior. The numerator polynomial will certainly play a role when finding the partial-fraction coefficients, but it plays a minor role in determining the essential features of the time-domain function.

Finding the coefficients

There are a couple of methods for finding the coefficients. (If you have previously studied Laplace methods in a diff. eq. math class, you have probably used one of these techniques.)

We will illustrate one method that works in most cases. Consider a third-order frequency-domain function. Assume that the denominator has already been factored.

$$F(s) = \frac{N(s)}{(s+P_2)(s+P_1)(s+P_0)}$$

To repeat, it doesn't really matter what form the numerator has — the essential features of the function are determined by the denominator.

The partial-fraction expression for the function is

$$\frac{N(s)}{(s+P_2)(s+P_1)(s+P_0)} = \frac{A_2}{(s+P_2)} + \frac{A_1}{(s+P_1)} + \frac{A_o}{(s+P_0)}$$

$$\frac{N(s)}{(s+P_2)(s+P_1)(s+P_o)} = \frac{A_2}{(s+P_2)} + \frac{A_1}{(s+P_1)} + \frac{A_o}{(s+P_o)}$$

Multiply both sides by $(s + P_2)\cdot(s + P_1)\cdot(s + P_0)$

$$N(s) = A_2(s+P_1)(s+P_o) + A_1(s+P_2)(s+P_o) + A_o(s+P_2)(s+P_1)$$

Evaluate the expression at the value of the P_2 pole, $s = -P_2$ or $(s + P_2 = 0)$. We see that the second and third terms on the right go to zero.

$$N(-P_2) = A_2(-P_2+P_1)(-P_2+P_o)$$

Solve for the coefficient.

$$A_2 = \frac{N(-P_2)}{(P_1-P_2)(P_o-P_2)}$$

The other coefficients are found in a similar manner. Go back to:

$$N(s) = A_2 \left(s + P_1\right) \left(s + P_o\right) + A_1 \left(s + P_2\right) \left(s + P_o\right) + A_o \left(s + P_2\right) \left(s + P_1\right)$$

Evaluate the expression at $s = - P_1$ or $(s + P_1 = 0)$. The first and third terms on the right go to zero.

$$N\left(-P_1\right) = A_1 \left(-P_1 + P_2\right) \left(-P_1 + P_o\right)$$

Solve for the coefficient: $A_1 = \dfrac{N\left(-P_1\right)}{\left(P_2 - P_1\right) \left(P_o - P_1\right)}$

Go back to the first expression above and evaluate it at $s = - P_o$,

$$N\left(-P_o\right) = A_o \left(-P_0 + P_2\right) \left(-P_o + P_1\right)$$

Solve for the final coefficient: $A_o = \dfrac{N\left(-P_o\right)}{\left(P_2 - P_o\right) \left(P_1 - P_o\right)}$

The method is straight-forward. And a bit tedious.

The partial-fraction version of the frequency domain expression is

$$F(s) = \frac{A_2}{(s+P_2)} + \frac{A_1}{(s+P_1)} + \frac{A_o}{(s+P_o)}$$

The inverse transformation can be done term-by-term to arrive at the time-domain result.

$$f(t) = A_2 e^{-P_2 t} + A_1 e^{-P_1 t} + A_o e^{-P_o t}$$

We will do a couple of simple numerical examples before trying circuits.

Example 1

Find the inverse transform of $F(s) = \dfrac{10}{s^2 + 7s + 10}$

The denominator is second order, so we expect two poles and thus two terms in the partial fraction result.

First, factor the denominator: $F(s) = \dfrac{10}{(s+5)(s+2)}$

Poles at $s = -3$, and $s = -7$.

The partial-fraction expansion is: $\dfrac{10}{(s+5)(s+2)} = \dfrac{A_1}{s+5} + \dfrac{A_o}{s+2}$

Multiply by $(s + 5)\cdot(s + 2)$

$$10 = A_1(s+2) + A_o(s+5)$$

Evaluate at $s = -5$: $10 = A_1(-3) + 0 \rightarrow A_1 = -10/3 = -3.33$.

Evaluate at $s = -2$: $10 = 0 + A_o(3) \rightarrow A_o = 10/3 = 3.33$.

The frequency-domain partial fraction result is: $F(s) = -\dfrac{3.33}{s+5} + \dfrac{3.33}{s+2}$

The corresponding time-domain result is: $f(t) = -3.33 \cdot e^{-5t} + 3.33 \cdot e^{-2t}$

Example 2

Find the inverse transform of $F(s) = \dfrac{s+4}{s^3 + 10s^2 + 21s}$

The denominator is third order, so we expect three poles and then three terms in the partial fraction result.

First, factor the denominator: $F(s) = \dfrac{s+4}{s(s+3)(s+7)}$

Zero at $s = -4$.

Poles are at $s = 0$, $s = -3$, and $s = -7$.

The partial-fraction expansion is: $\dfrac{s+4}{s(s+3)(s+7)} = \dfrac{A_2}{s} + \dfrac{A_1}{s+3} + \dfrac{A_o}{s+7}$

Multiply by $s \cdot (s+3) \cdot (s+7)$:

$$s + 4 = A_2(s+3)(s+7) + A_1 s(s+7) + A_o s(s+3)$$

Evaluate at $s = 0$: $4 = A_2(3)(7) + 0 + 0 \rightarrow A_2 = 4/21 = 0.190$.

Evaluate at $s = -3$: $1 = 0 + A_1(-3)(4) + 0 \rightarrow A_1 = -1/12 = -0.083$.

Evaluate at $s = -7$: $-3 = 0 + 0 + A_o(-7)(-4) \rightarrow A_o = -3/28 = -0.107$.

Then the frequency-domain partial fraction result is:

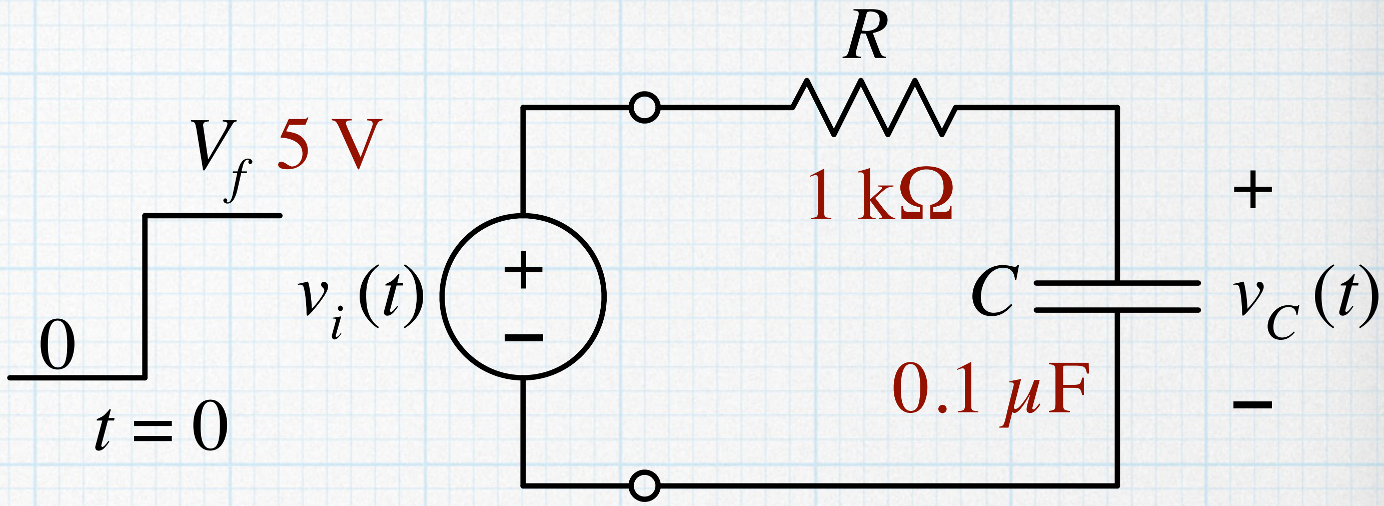
$$F(s) = \frac{0.190}{s} - \frac{0.083}{s+3} - \frac{0.107}{s+7}$$

The corresponding time-domain result is

$$f(t) = 0.190 \cdot u(t) - 0.083 \cdot e^{-3t} - 0.107 \cdot e^{-7t}$$

Example 3

Now we can apply the method to the step input RC circuit. Values for components and source voltages have been added so that we can begin to get a feel for the types of numbers involved.



$RC = 0.1$ msec

As we saw previously, the frequency-domain expression for the capacitor voltage is:

$$V_C(s) = \frac{V_f/RC}{s\left(s+\frac{1}{RC}\right)} = \frac{\frac{5\ \text{V}}{0.1\ \text{msec}}}{s\left(s+\frac{1}{0.1\ \text{msec}}\right)} = \frac{50000\ \text{V/sec}}{s\left(s+10^4\ \text{sec}^{-1}\right)}$$

The partial fraction expansion is:

Poles are at $s = 0$ and $s = -\frac{1}{RC} = -10^4\ \text{sec}^{-1}$.

$$\frac{V_f/RC}{s\left(s+\frac{1}{RC}\right)} = \frac{A_1}{s} + \frac{A_o}{\left(s+\frac{1}{RC}\right)}$$

$$\frac{V_f/_{RC}}{s\left(s+\frac{1}{RC}\right)} = \frac{A_1}{s} + \frac{A_o}{\left(s+\frac{1}{RC}\right)}$$

Multiply both sides by $s{\cdot}(s + 1/RC)$:

$$\frac{V_f}{RC} = A_1\left(s+\frac{1}{RC}\right) + A_o s$$

Evaluate at $s = -\frac{1}{RC}$: $\frac{V_f}{RC} = 0 + A_o\left(-\frac{1}{RC}\right) \rightarrow A_o = -V_f$

Evaluate at $s = 0$: $\frac{V_f}{RC} = A_1\left(\frac{1}{RC}\right) + 0 \rightarrow A_1 = V_f$

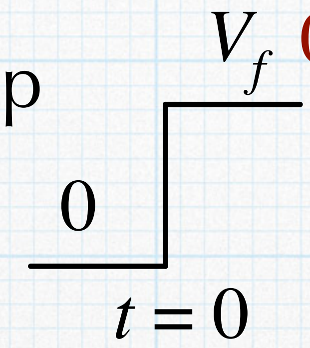
Then: $V_C(s) = \frac{V_f}{s} - \frac{V_f}{s+\frac{1}{RC}}$

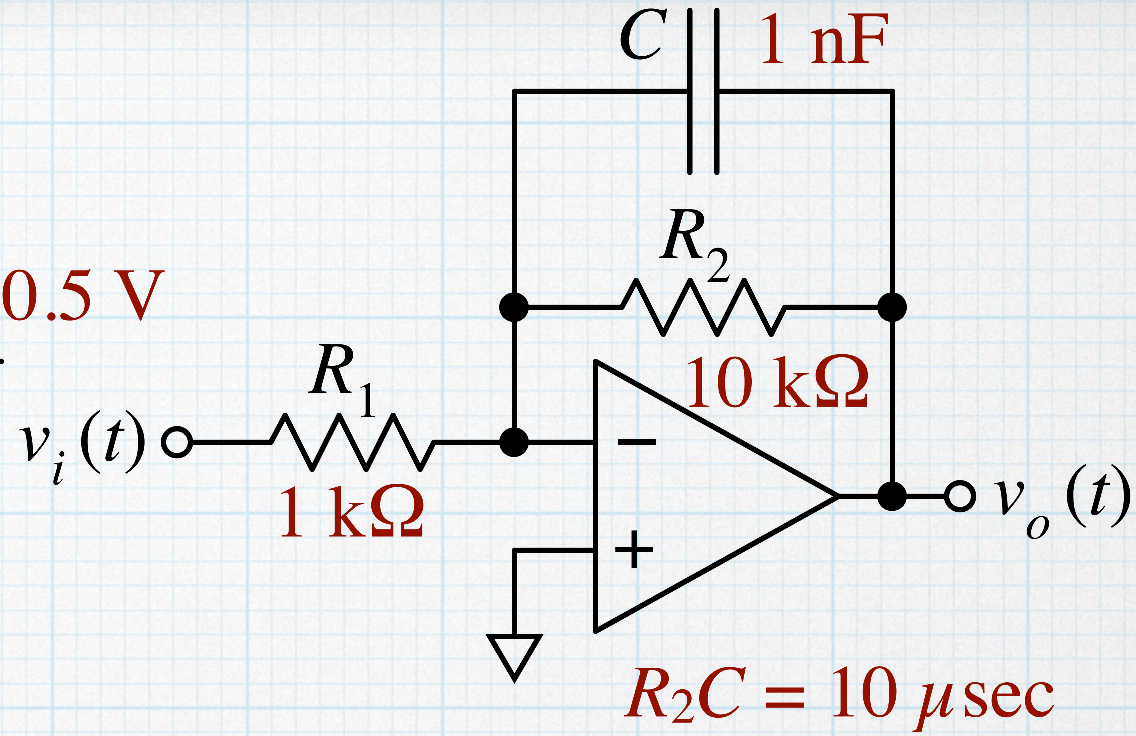
Finally: $v_C(t) = V_f \cdot u(t) - V_f \cdot \exp\left(-\frac{t}{RC}\right)$

Matches EE 201 result exactly.

Example 4

Now let's try the previously worked op amp circuit with step input.





As we saw previously, the frequency-domain expression for the output voltage is:

$$V_o(s) = -\frac{\frac{V_f}{R_1C}}{s\left(s+\frac{1}{R_2C}\right)} = \frac{\frac{0.1\text{ V}}{10\,\mu\text{sec}}}{s\left(s+\frac{1}{10\,\mu\text{sec}}\right)} = \frac{10^4\text{ V/sec}}{s\left(s+10^5\text{ sec}^{-1}\right)}$$

We can already see that this will be virtually identical to the previous *RC* example. This time, let's work with number instead of symbols.

$$\frac{10^4\text{ V/sec}}{s\left(s+10^5\text{ sec}^{-1}\right)} = \frac{A_1}{s} + \frac{A_o}{\left(s+10^5\text{ sec}^{-1}\right)}$$

$$\frac{10^4\ \text{V/sec}}{s\left(s+10^5\ \text{sec}^{-1}\right)} = \frac{A_1}{s} + \frac{A_o}{s+10^5\ \text{sec}^{-1}}$$

Following the same steps:

$$10^4\ \text{V/sec} = A_1\left(s+10^5\ \text{sec}^{-1}\right) + A_o s$$

Evaluate at $s = -10^5\ \text{sec}^{-1}$: $10^4\ \text{V/sec} = 0 + A_o\left(-10^5\ \text{sec}^{-1}\right) \rightarrow A_o = -\ 10\ \text{V}$

Evaluate at $s = 0$: $10^4\ \text{V/sec} = A_1\left(10^5\ \text{sec}^{-1}\right) + 0 \rightarrow A_1 = 10\ \text{V}$

Then: $V_C(s) = \dfrac{10\ \text{V}}{s} - \dfrac{10\ \text{V}}{s+10^5\ \text{sec}^{-1}}$

Finally: $v_C(t) = (10\ \text{V})\cdot u(t) - (10\ \text{V})\cdot\exp\left[-\left(10^5\ \text{sec}^{-1}\right)t\right]$

$$= (10\ \text{V})\cdot u(t) - (10\ \text{V})\cdot\exp\left(-\frac{t}{10\ \mu\text{sec}}\right)$$

Even though the simple *RC* and the op amp appear to be very different circuits, in the frequency domain, they are essentially identical.

Repeated roots – special case

We should consider the special case a frequency-domain expression having repeated poles. This requires a bit of extra work. We could argue the that extra work is not worth it, because we will (almost) never encounter repeated poles in a real circuit. Obtaining repeated poles would require perfect matching of components, which, given typical component tolerances, would probably only happen by accident.

But we should cover the repeated pole situation. It is interesting from an academic point of view, and it provides a dividing line between functions have distinct real poles and those having complex poles. (In EE 201 terms, repeated poles corresponds to critical damping in an *RLC* step response, which is the dividing line between over-damped and under-damped responses.)

As an example to illustrate the method for handling repeated roots, consider the function $\frac{s}{(s+3)^2}$. This is a second-order function and has two roots, both of which are at $s = -3$. If we try to expand this as $\frac{A_1}{s+3} + \frac{A_o}{s+3}$, the problem in choosing A_1 and A_o is immediately apparent.

Instead, the expansion should consist of terms that have various powers of the repeated pole. For our example:

$$\frac{s}{(s+3)^2} = \frac{A_1}{s+3} + \frac{A_o}{(s+3)^2}.$$

With that starting point, we can proceed as before. However, we will not be able to simply plug in values of the poles to find the coefficients. Instead we will arrive a set of simultaneous equations relating the coefficients. Solving the set can range from trivial to tedious, depending on the order of the starting function.

Subtle point: The units of the coefficients in the in the expansion will be different. For example, if s is frequency (sec^{-1}) in the above expression, then the units of the left-hand side is sec. Then on the right-hand side, A_1 must be dimensionless (like a gain) and A_o will be a frequency (sec^{-1}). In a typical math problem, where we are dealing simply with numbers, we never worry about the "meaning" of the coefficients, but in a real, physical system, like a circuit, we must know what the symbols represent and keep track of the corresponding units. (Also, if the above expression represents a real system, we should includes the units with the numbers, "$3\ \text{sec}^{-1}$" and not just "3".)

$$\frac{s}{(s+3)^2} = \frac{A_1}{s+3} + \frac{A_o}{(s+3)^2}.$$

Continuing on with the example, we can multiply both sides by $(s + 3)^2$,

$$s = A_1(s+3) + A_o$$

$$= A_1 s + (3A_1 + A_o)$$

Matching coefficients of the powers of s on the left and right gives two equations

$A_1 = 1$ and $3A_1 + A_o = 0$.

Solving these falls into the trivial class: $A_1 = 1$ and $A_o = -3$.

The partial fraction version of the frequency-domain function is

$$\frac{1}{s+3} - \frac{3}{(s+3)^2}$$

That leaves us with the final question: What is inverse transform of the second term. Answer: it is a "damped ramp", $t{\cdot}\exp(-at) \leftrightarrow (s + a)^{-2}$. So the corresponding time-domain function is: $f(t) = \exp(-t) - 3t{\cdot}\exp(-3t)$.

Example 5

Find the time-domain function corresponding to the frequency-domain function:

$$F(s) = \frac{3\,(s+4)^2}{(s+5)\left(s^2+7s+10\right)}.$$

Factoring the quadratic in the denominator: $s^2 + 2s + 10 = (s+5)\cdot(s+2)$, we see that the function actually has repeated poles. So, rewriting:

$$F(s) = \frac{3\,(s+4)^2}{(s+5)^2\,(s+2)}.$$

The partial fraction expansion will be:

$$\frac{3\,(s+4)^2}{(s+5)^2\,(s+2)} = \frac{A_2}{(s+5)^2} + \frac{A_1}{s+5} + \frac{A_o}{s+2}.$$

Multiply both sides by $(s+5)^2(s+2)$:

$$3(s+4)^2 = A_2\,(s+2) + A_1\,(s+5)(s+2) + A_o\,(s+5)^2$$

$$3(s+4)^2 = A_2(s+2) + A_1(s+5)(s+2) + A_o(s+5)^2$$

Continue with the tedious algebra — expand everything out:

$$3(s^2+8s+16) = A_2(s+2) + A_1(s^2+7s+10) + A_o(s^2+10\text{s}+25)$$

Gather together similar powers:

$$3s^2+24s+48 = (A_1+A_o)s^2 + (A_2+7A_1+10A_o)s + (2A_2+10A_1+25A_o)$$

Match the coefficients for similar powers, resulting in 3 equations in 3 unknowns, A_2, A_1, and A_o.

$$A_1 + A_o = 3 \;;\; A_2 + 7A_1 + 10A_o = 24 \;;\; 2A_2 + 10A_1 + 25A_o = 48$$

Use your favorite method to solve: (Typically, I default to Wolfram Alpha.)

$$A_2 = -1 \;;\; A_1 = 1.67 \;;\; A_o = 1.33$$

The complete partial-fraction expansion is: $F(s) = -\dfrac{1}{(s+5)^2} + \dfrac{1.67}{s+5} + \dfrac{1.33}{s+2}$

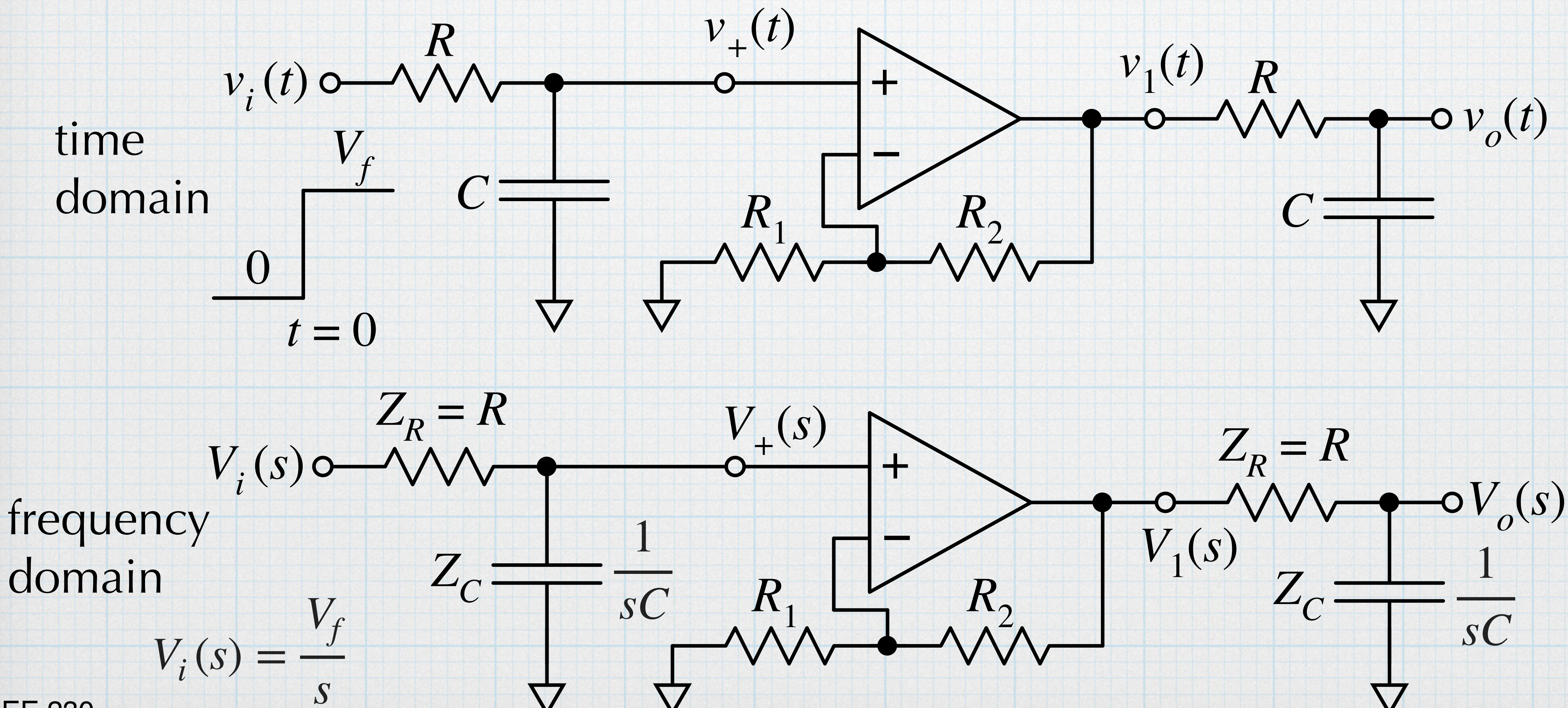
Going back to the time domain:

$$f(t) = -t\cdot\exp(-5t) + 1.67\cdot\exp(-5t) + 1.33\cdot\exp(-2t)$$

Example 6

We should try a circuit that has repeated poles. The circuit below consists of three sections, an *RC* voltage divider at the input, an non-inverting op amp, and an *RC* section at the output, identical to the first. The source voltage has a step change, starting at 0 and stepping up to V_f.

The isolation provided by the ideal input and output resistances of the amp allow us to treat the circuit as three independent pieces which are then cascaded to give the total response.



Voltage divider at the input: $V_+(s) = \dfrac{\frac{1}{sC}}{R + \frac{1}{sC}} v_i = \dfrac{\frac{1}{RC}}{s + \frac{1}{RC}} V_i(s)$

Gain of the non-inverting: $V_1(s) = \left(1 + \dfrac{R_2}{R_1}\right) V_+(s) = G_o V_+(s)$

Voltage divider at the output: $V_o(s) = \dfrac{\frac{1}{sC}}{R + \frac{1}{sC}} V_1(s) = \dfrac{\frac{1}{RC}}{s + \frac{1}{RC}} V_1(s)$

Putting the three pieces together: $V_o(s) = \dfrac{G_o \left(\frac{1}{RC}\right)^2}{\left(s + \frac{1}{RC}\right)^2} V_i(s)$

Inserting the source: $V_o(s) = \dfrac{G_o \cdot \left(\frac{1}{RC}\right)^2 \cdot V_f}{s\left(s + \frac{1}{RC}\right)^2} = \dfrac{K}{s(s+a)^2}$

where new names $a = 1/RC$ and $K = G_o V_f/(RC)^2 = G_o V_f a^2$ and are introduced to help tidy up the impending math.

Do the partial fraction expansion, using the trick for repeated poles:

$$V_o(s) = \frac{K}{s(s+a)^2} = \frac{A_2}{(s+a)^2} + \frac{A_1}{s+a} + \frac{A_o}{s}$$

Multiply by $s(s+a)^2$: $K = A_2 s + A_1 s(s+a) + A_o(s+a)^2$

Expand and gather coefficients: $K = (A_1 + A_o)s^2 + (A_2 + A_1 a + 2A_o a)s + A_o a^2$

This is one closer to a trivial case. Comparing terms:

$$K = A_o a^2 \rightarrow A_o = \frac{K}{a^2} = G_o V_f$$

$$A_1 + A_o = 0 \rightarrow A_1 = -A_o \rightarrow A_1 = -G_o V_f$$

$$A_2 + A_1 a + 2A_o a = 0 \rightarrow A_2 = -A_1 a - 2A_o a \rightarrow A_2 = -\frac{G_o V_f}{RC}$$

$$V_o(s) = -\frac{\frac{G_o V_f}{RC}}{\left(s + \frac{1}{RC}\right)^2} - \frac{G_o V_f}{s + \frac{1}{RC}} + \frac{G_o V_f}{s}$$

$$v_o(t) = -G_o V_f \frac{t}{RC} \exp\left(-\frac{t}{RC}\right) - G_o V_f \exp\left(-\frac{t}{RC}\right) + G_o V_f \cdot \mathrm{u}(t)$$

Imaginary and complex roots – sinusoids

Now, we consider that situation when a circuit is being driving by a sinusoidal source. Introducing a sinusoid (sine or cosine) into a frequency-domain expression necessarily creates complex poles (and maybe complex zeros). This is a continuation of the EE 201 notion that sinusoids and complex numbers are inextricably linked together.

The partial fraction approach with complex poles is essentially the same as with real poles. The difference is in the complex algebra that reaults. In addition to the slightly messier complex algebra, one or two extra steps may be needed in order to put the frequency-domain and time-domain expressions into a recognizable form.

We should remind ourselves of some basic tools needed when performing complex algebra. Then next slide lists a few key details for complex math. (If your complex number skills are shaky, it would probably be a good idea to review basic complex analysis before diving too deeply into the subsequent material. The EE 201 notes contain a few slides covering basic complex algebra and the proper interpretation of complex numbers in circuit analysis.)

Complex number math & sinusoids

Real/imaginary

Magnitude/phase

$$K = a + jb = Me^{j\theta}$$

$$a = M\cos\theta \qquad M = \sqrt{a^2 + b^2}$$

$$b = M\sin\theta \qquad \theta = \arctan\left(\frac{b}{a}\right)$$

Complex conjugate

$$K = a + jb \rightarrow K^* = a - jb$$

$$K = Me^{j\theta} \rightarrow K^* = Me^{-j\theta}$$

Euler

$$e^{j\theta} = \cos\theta + j\sin\theta \qquad \cos\theta = \frac{1}{2}\left[e^{j\theta} + e^{-j\theta}\right]$$

$$Me^{j\theta} = M\cos\theta + jM\sin\theta \qquad \sin\theta = \frac{-j}{2}\left[e^{j\theta} - e^{-j\theta}\right]$$

Common identity $\frac{1}{j} = -j$

Laplace

$$\cos\omega t \leftrightarrow \frac{s}{s^2 + \omega^2} \qquad \sin\omega t \leftrightarrow \frac{\omega}{s^2 + \omega^2} \qquad e^{\pm j\omega t} \leftrightarrow \frac{1}{s \mp j\omega}$$

Useful trig identity $A\cos\omega t + B\sin\omega t = M\cos(\omega t - \theta)$

$$M = \sqrt{A^2 + B^2} \qquad \theta = \arctan\left(\frac{B}{A}\right)$$

Example 7 – inverse transform with imaginary poles

The best way to see how to handle complex numbers is do some examples. Consider the frequency-domain function

$$F(s) = \frac{-18}{(s+3)\left(s^2+9\right)}$$

(We can imagine how a sinusoidal source would result in the $s^2 + 9$ factor in the denominator.)

Factoring the denominator: $F(s) = \dfrac{-18}{(s+3)\left(s+j3\right)\left(s-j3\right)}$

Two of the poles are imaginary, $P_2 = -j3$ and $P_3 = +j3$. Note that the poles are complex conjugates. This is always the case with frequency-domain functions that come from real circuits — complex poles always come in complex-conjugate pairs. This is important for ensuing steps.

Expand the function into partial fractions

$$\frac{-18}{(s+3)\left(s+j3\right)\left(s-j3\right)} = \frac{A_2}{s+3} + \frac{A_1}{s+j3} + \frac{A_o}{s-j3}$$

$$\frac{-18}{(s+3)\left(s+j3\right)\left(s-j3\right)} = \frac{A_2}{s+3} + \frac{A_1}{s+j3} + \frac{A_o}{s-j3}$$

Multiply by $(s+3)(s+j3)(s-j3)$

$$-18 = A_2\left(s+j3\right)\left(s-j3\right) + A_1(s+3)\left(s-j3\right) + A_o(s+3)\left(s+j3\right)$$

Evaluate at each of the pole values:

At $s = -3$:

$$-18 = A_2\left(-3+j3\right)\left(-3-j3\right) + 0 + 0 \rightarrow A_2 = \frac{-18}{\left(-3+j3\right)\left(-3-j3\right)} = \frac{-18}{18} = -1$$

At $s = -j3$:

$$-18 = 0 + A_1\left(-j3+3\right)\left(-j6\right) + 0 \rightarrow A_1 = \frac{-18}{-18-j18} = \frac{1}{1+j1} = 0.707 \cdot e^{-j45^{\circ}}$$

At $s = +j3$:

$$-18 = 0 + 0 + A_o\left(j3+3\right)\left(j6\right) \rightarrow A_o = \frac{-18}{-18+j18} = \frac{1}{1-j1} = 0.707 \cdot e^{j45^{\circ}}$$

Note that A_1 and A_o are complex conjugates. This also will always be true — the coefficients for the complex conjugate poles will be complex conjugates as well. Meaning that if we calculate one, we know the other automatically.

The partial fraction expansion of the function is

$$F(s) = -\frac{1}{s+3} + \frac{0.5 - j0.5}{s + j3} + \frac{0.5 + j0.5}{s - j3}$$

Converting back to the time domain:

$$f(t) = -\exp(-3t) + \left(0.5 - j0.5\right) \exp\left(j3t\right) + \left(0.5 + j0.5\right) \exp\left(-j3t\right)$$

Lots of complex numbers. What does it all mean? Re-arrange a bit:

$$f(t) = -e^{-3t} + 0.5\left[e^{j3t} + e^{-j3t}\right] - j0.5\left[e^{j3t} - e^{-j3t}\right]$$

Use Euler's relations to convert the complex exponentials to sine & cosine:

$$f(t) = e^{-3t} + \cos(3t) - \sin(3t)$$

Then use the $\cos\theta + \sin\theta$ identity to combine the two sinusoids:

$$f(t) = e^{-3t} + 1.414 \cdot \cos\left(3t + 45^\circ\right)$$

Complex conjugates poles in the frequency domain will always lead back to sinusoids in the time domain. And we should expect real coefficients in the final result.

Rather than converting the simple partial fraction form of the frequency-domain to the time domain and then slashing around with complex numbers to simply the time-domain form, we could do the slashing around first. After find the coefficient of the PF expansion,

$$F(s) = -\frac{1}{s+3} + \frac{0.5 - j0.5}{s+j3} + \frac{0.5 + j0.5}{s-j3}$$

Combine the 2nd and 3rd terms using common denominators.

$$F(s) = -\frac{1}{s+3} + \frac{(0.5 - j0.5)(s-j3)}{(s+j3)(s-j3)} + \frac{(0.5 + j0.5)(s+j3)}{(s-j3)(s+j3)}$$

$$= -\frac{1}{s+3} + \frac{0.5s - j0.5s - j1.5 - 1.5 + 0.5s + j0.5s + j1.5 - 1.5}{s^2+9}$$

After simplifying the numerator, we are left with

$$F(s) = -\frac{1}{s+3} + \frac{s-3}{s^2+9} = -\frac{1}{s+3} + \frac{s}{s^2+9} - \frac{3}{s^2+9}$$

Each term is familiar from our small Laplace table.

$$f(t) = e^{-3t} + \cos(3t) - \sin(3t)$$ It's the same.

Now that we have some feel for what will happen when there are complex conjugate poles, we can try yet a third method to do find the inverse. Since we know that in the end, there will be sines and cosines in the time-domain functions, we can tilt things in that direction by using a partial fraction expansion that starts with sine and cosine terms.

$$F(s) = \frac{-18}{(s+3)\left(s^2+9\right)} = \frac{B_2}{s+3} + \frac{B_1 s}{s^2+9} + \frac{B_o \cdot 3}{s^2+9}$$

We recognize the 2nd and 3rd as being cosine and sine. If the can find the correct coefficients, we can avoid some of the complex math.

Do the usual:

$$-18 = B_2\left(s^2+9\right) + B_1 s(s+3) + 3B_o(s+3)$$

$$= \left(B_2 + B_1\right)s^2 + \left(3B_1 + 3B_o\right)s + \left(9B_2 + 9B_o\right)$$

Matching coefficients for the different powers, we arrive at three equations in the 3 unknowns: $B_2 + B_1 = 0$; $B_1 + B_o = 0$; $B_2 + B_o = -2$. This set falls into the trivial category: $B_2 = -1, B_1 = 1, B_o = -1$, but use a solver if needed.

$$F(s) = -\frac{1}{s+3} + \frac{s}{s^2+9} - \frac{3}{s^2+9}$$

Boom. Convert back.

Example 8 – complex poles

Sinusoidal problems may have poles that are complex poles, not just imaginary. Complex poles lead to damped sinusoids. With complex poles, the basic approach is the same, but (as is always the case), the complex math may be tedious. Consider the frequency-domain function below:

$$F(s) = \frac{-10\,(s+4)}{s\left(s^2+8s+25\right)}$$

If we finish factoring the poles, we obtain a complex-conjugate pair.

$$F(s) = \frac{-10\,(s+4)}{s\left(s+4+j3\right)\left(s+4-j3\right)} = \frac{-10\,(s+4)}{s\,(s+P)\,(s+P^*)}$$

(We use $P = 4 + j3$ and $P^* = 4 - j3$ to denote the values and help reduce math mess.) Expanding the function in the usual fashion.

$$\frac{-10\,(s+4)}{s\,(s+P)\,(s+P^*)} = \frac{A_2}{s} + \frac{A_1}{s+P} + \frac{A_o}{s+P^*}$$

On the right, we make use of our earlier observation that the coefficients will be complex conjugates.

$$\frac{-10\,(s+4)}{s\,(s+P)\,(s+P^*)} = \frac{A_2}{s} + \frac{A_1}{s+P} + \frac{A_o}{s+P^*}$$

Multiply by the left-side denominator:

$$-10\,(s+4) = A_2\,(s+P)\,(s+P^*) + A_1 s\,(s+P^*) + A_o s\,(s+P)$$

Evaluate at $s = 0$:

$$-10\,(4) = A_2\left(25\right) \rightarrow A_2 = -\,1.6$$

Evaluate at $s = -P = -4 - j3$:

$$-10\left(-4 - j3 + 4\right) = A_1\left(-4 - j3\right)\left(-4 - j3 + 4 - j3\right)$$

$$A_1 = \frac{-10\left(-j3\right)}{\left(-4 - j3\right)\left(-j6\right)} = \frac{5}{4 + j3} = \frac{5}{5e^{j36.9^\circ}} = 1e^{-j36.9^\circ} = 0.8 - j0.6$$

We could find the final coefficient by evaluating the above expression at $s = -P^* = -4 + j3$. Or, we use a more expedient approach by recalling that the coefficients for complex poles must themselves be complex conjugates.

$$A_o = A_1^* = 0.8 + j0.6 = 1e^{j36.9^\circ}$$

Putting it all together:

$$F(s) = \frac{-1.6}{s} + \frac{0.8 - j0.6}{s + (4 + j3)} + \frac{0.8 + j0.6}{s + (4 - j3)}$$

We can plow ahead, transforming back to time domain directly and then manipulating the results there. Or we could try the technique of combining the two complex conjugate terms in the frequency domain and massaging them into convenient form before transforming. Let's plow ahead. We recognize the terms as a step and two exponentials.

$$f(t) = -\,1.6 \cdot \mathrm{u}(t) + (0.8 - j0.6)\, e^{-(4 + j3)t} + (0.8 + j0.6)\, e^{-(4 - j3)t}$$

Re-arrange and gather together related terms.

$$f(t) = -\,1.6 \cdot \mathrm{u}(t) + 0.8 \cdot e^{-4t} \left(e^{j3t} + e^{-j3t}\right) + j0.6 \cdot e^{-4t} \left(e^{j3t} - e^{-j3t}\right)$$

Use Euler to convert complex exponentials to sinusoids

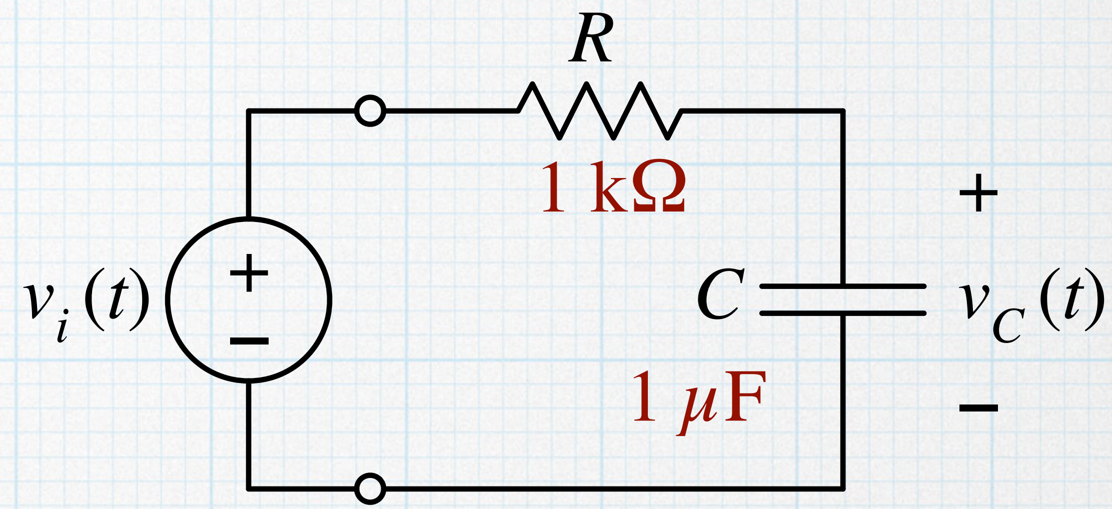
$$f(t) = -\,1.6 \cdot \mathrm{u}(t) + 1.6 \cdot e^{-4t} \cos(3t) - 1.2 \cdot e^{-4t} \sin(3t)$$

Finally, if desired, use the trig identity to combine the cosine and sine

$$f(t) = -\,1.6 \cdot \mathrm{u}(t) + 2 \cdot e^{-4t} \cos\left(3t + 36.9°\right)$$

Example 9 – simple RC with sinusoidal source

This is the basic *RC* circuit that we have seen before. (In Example 3 above, we found the step response.) Previously, we calculated the frequency-domain function for the capacitor voltage. ("Example 2 in "Laplace Circuits" notes.) As seen previously, the two impedance from a simple voltage divider.



$$v_i(t) = V_A \cdot \cos(\omega t)$$

$V_A = 5$ V $\omega = 1000$ rad/s

The Laplace expression for the source is $V_i(s) = \dfrac{V_A \cdot s}{s^2 + \omega^2}$

$$V_C(s) = \left(\frac{\frac{1}{RC}}{s + \frac{1}{RC}}\right)\left(\frac{V_A \cdot s}{s^2 + \omega^2}\right) = \frac{\frac{V_A}{RC} \cdot s}{\left(s + \frac{1}{RC}\right)\left(s^2 + \omega^2\right)}$$

This is identical to Example 7. The partial-fraction expansion is

$$\frac{\frac{V_A}{RC} \cdot s}{\left(s + \frac{1}{RC}\right)\left(s^2 + \omega^2\right)} = \frac{A_2}{s + \frac{1}{RC}} + \frac{A_1}{s + j\omega} + \frac{A_2}{s - j\omega}$$

$$\frac{\frac{V_A}{RC}\cdot s}{\left(s+\frac{1}{RC}\right)\left(s^2+\omega^2\right)} = \frac{A_2}{s+\frac{1}{RC}} + \frac{A_1}{s+j\omega} + \frac{A_o}{s-j\omega}$$

Do the usual math:

$$\frac{V_A}{RC}\cdot s = A_2\left(s^2+\omega^2\right) + A_1\left(s+\frac{1}{RC}\right)\left(s-j\omega\right) + A_o\left(s+\frac{1}{RC}\right)\left(s+j\omega\right)$$

Since all poles are distinct, we can find the coefficients one by one.

Evaluate at $s = -1/RC$:

$$-\frac{V_A}{(RC)^2} = A_2\left[\left(\frac{1}{RC}\right)^2 + \omega^2\right] + 0 + 0 \rightarrow A_2 = -\frac{V_A}{1+(\omega RC)^2} = -\,2.5\,\text{V}$$

Evaluate at $s = -j\omega$:

$$\frac{V_A}{RC}\left(-j\omega\right) = A_1\left(-j\omega+\frac{1}{RC}\right)\left(-j2\omega\right) \rightarrow A_1 = \frac{V_A\left(1+j\omega RC\right)}{2\left[1+(\omega RC)^2\right]} = 1.25\,\text{V}\left(1+j1\right)$$

Evaluate at $s = +j\omega$:

$$\frac{V_A}{RC}\left(j\omega\right) = A_o\left(j\omega+\frac{1}{RC}\right)\left(j2\omega\right) \rightarrow A_o = \frac{V_A\left(1-j\omega RC\right)}{2\left[1+(\omega RC)^2\right]} = 1.25\,\text{V}\left(1-j1\right)$$

In looking at the form of the coefficients, we note once again that $A_o = A_1^*$. Also we can combine some details to write the frequency-domain expression as

$$V_C(s) = \frac{V_A}{1+(\omega RC)^2}\left[\frac{-1}{s+\frac{1}{RC}} + \frac{0.5\left(1+j\omega RC\right)}{s+j\omega} + \frac{0.5\left(1-j\omega RC\right)}{s-j\omega}\right]$$

All the terms represent exponentials, so conversion to time is easy.

$$v_C(t) = \frac{V_A}{1+(\omega RC)^2}\left[-e^{-\frac{t}{RC}} + 0.5\left(1+j\omega RC\right)e^{-j\omega t} + 0.5\left(1-j\omega RC\right)e^{+j\omega t}\right]$$

As we have done previously, re-group similar terms.

$$v_C(t) = \frac{V_A}{1+(\omega RC)^2}\left[-e^{-\frac{t}{RC}} + 0.5\left(e^{+j\omega t}+e^{-j\omega t}\right) - j0.5\left(\omega RC\right)\left(e^{+j\omega t}-e^{-j\omega t}\right)\right]$$

Using Euler:

$$v_C(t) = \frac{V_A}{1+(\omega RC)^2}\left[-e^{-\frac{t}{RC}} + \cos\omega t + (\omega RC)\sin\omega t\right]$$

Using the trig identity to combine the cosine and sine

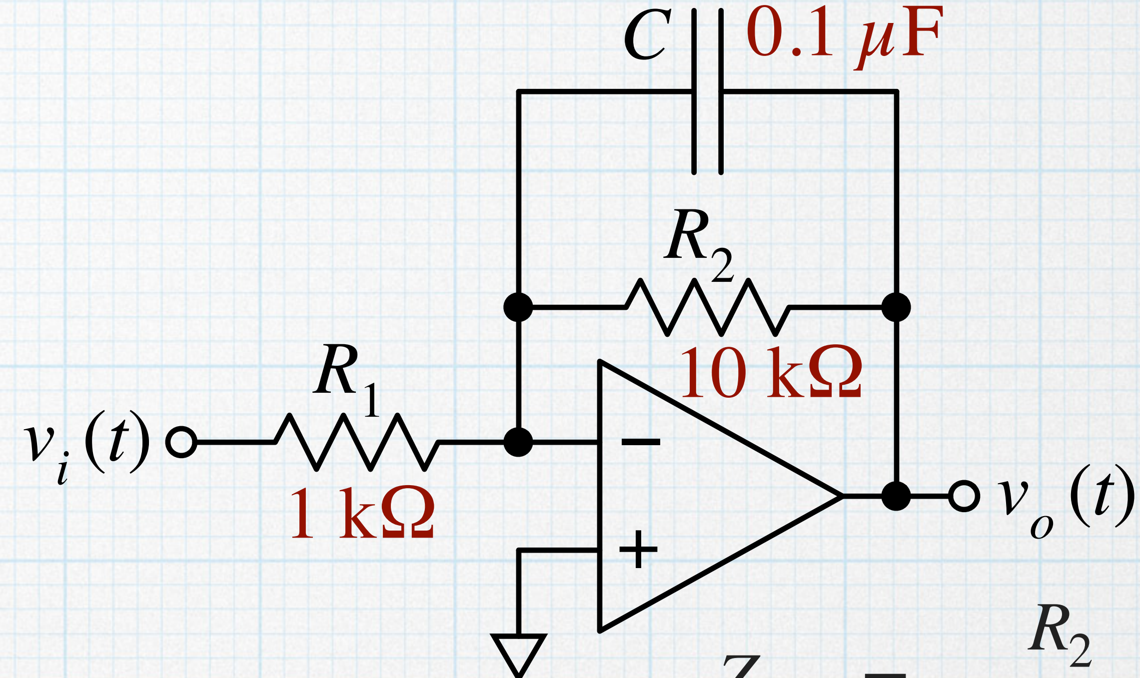
$$v_C(t) = V_1 \cdot e^{-\frac{t}{RC}} + V_2 \cdot \cos(\omega t - \theta)$$

$$V_1 = \frac{-V_A}{1+(\omega RC)^2} = -2.5\ \text{V} \quad V_2 = \frac{V_A}{\sqrt{1+(\omega RC)^2}} = 3.54\ \text{V} \quad \theta = \arctan(\omega RC) = 45^\circ$$

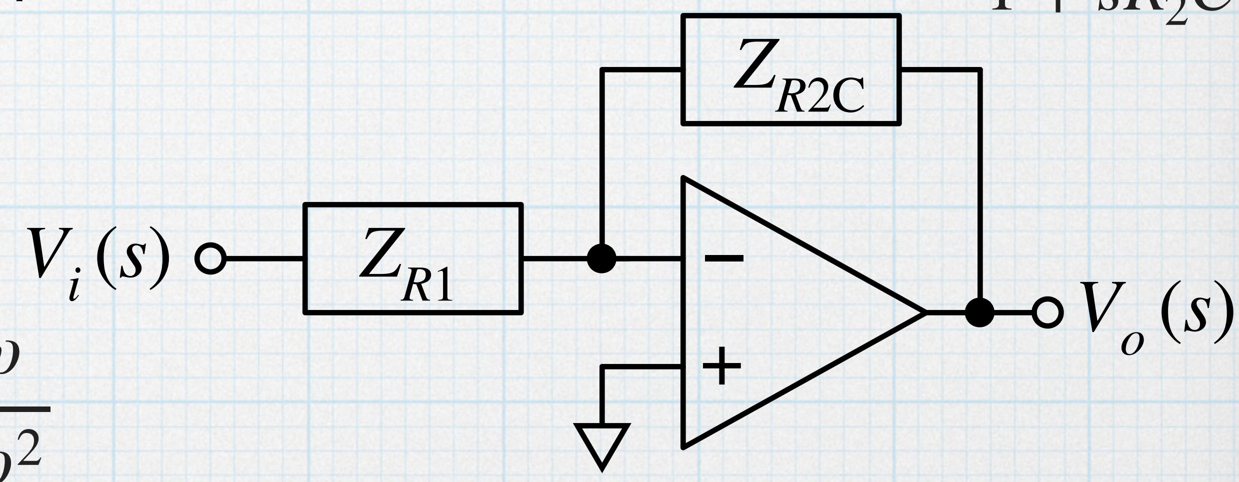
Example 10 – op amp with sinusoidal source

This is the same op amp with RC feedback that we have seen before. (In Example 4 above, in which found the step response, and in Example r in "Laplace Circuits" notes where we found the frequency domain expression for the output voltage.)

This time, let's go a with sine function for the source, to see how it works in the analysis.



$$Z_{R2C} = \frac{R_2}{1 + sR_2C}$$



$$v_i(t) = V_A \cdot \sin(\omega t) \rightarrow V_i(s) = \frac{V_A \cdot \omega}{s^2 + \omega^2}$$

$V_A = 0.5$ V $\quad \omega = 1000$ rad/s

$$V_o(s) = \left(-\frac{Z_{R2C}}{Z_{R1}}\right) V_i(s) = \left(-\frac{\frac{R_2}{R_1}}{1 + sR_2C}\right)\left(\frac{V_A \cdot \omega}{s^2 + \omega^2}\right) = \frac{-\frac{V_A \omega}{R_1 C}}{\left(s + \frac{1}{R_2 C}\right)\left(s^2 + \omega^2\right)}$$

For fun and variety, let's use the partial-fraction approach where we try to cast the function directly into terms that look like sines and cosines, (since we know that sines and cosines will be showing up).

$$V_o(s) = \frac{-\frac{V_A \omega}{R_1 C}}{\left(s + \frac{1}{R_2 C}\right)\left(s^2 + \omega^2\right)} = \frac{A_2}{s + \frac{1}{R_2 C}} + \frac{A_1 s}{s^2 + \omega^2} + \frac{A_o \omega}{s^2 + \omega^2}$$

The approach is familiar — we will leave out the comments.

$$-\frac{V_A \omega}{R_1 C} = A_2\left(s^2 + \omega^2\right) + A_1 s\left(s + \frac{1}{R_2 C}\right) + A_o \omega\left(s + \frac{1}{R_2 C}\right)$$

$$= \left(A_2 + A_1\right)s^2 + \left(\frac{A_1}{R_2 C} + A_o \omega\right)s + \left(A_2 \omega^2 + \frac{A_o \omega}{R_2 C}\right)$$

$$A_2 + A_1 = 0 \qquad \frac{A_1}{R_2 C} + A_o \omega = 0 \qquad -\frac{V_A \omega}{R_1 C} = A_2 \omega^2 + \frac{A_o \omega}{R_2 C}$$

$$A_2 = \frac{-\left(\frac{R_2}{R_1}V_A\right)\left(\omega R_2 C\right)}{1 + \left(\omega R_2 C\right)^2} \qquad A_1 = \frac{\left(\frac{R_2}{R_1}V_A\right)\left(\omega R_2 C\right)}{1 + \left(\omega R_2 C\right)^2} \qquad A_o = \frac{-\left(\frac{R_2}{R_1}V_A\right)}{1 + \left(\omega R_2 C\right)^2}$$

$$V_o(s) = \frac{\frac{R_2}{R_1} V_A}{1 + (\omega R_2 C)^2} \left[-\frac{\omega R_2 C}{s + \frac{1}{RC}} + \frac{(\omega R_2 C)\, s}{s^2 + \omega^2} - \frac{\omega}{s^2 + \omega^2} \right]$$

Converting back to time.

$$v_o(t) = \frac{\frac{R_2}{R_1} V_A}{1 + (\omega R_2 C)^2} \left[-(\omega R_2 C)\, e^{-\frac{t}{R_2 C}} + (\omega R_2 C) \cos \omega t - \sin \omega t \right]$$

Using the trig identity to combine the cosine and sine

$$v_C(t) = V_1 \cdot e^{-\frac{t}{R_2 C}} + V_2 \cdot \cos(\omega t - \theta)$$

$R_2/R_1 = 10$

$\omega R_2 C = 1$

$$V_1 = -\frac{\frac{R_2}{R_1} V_A (\omega R_2 C)}{1 + (\omega R_2 C)^2} = 2.5 \text{ V}$$

$$V_2 = \frac{\frac{R_2}{R_1} V_A}{\sqrt{1 + (\omega R_2 C)^2}} = 3.54 \text{ V}$$

$$\theta = \arctan\left(\frac{-1}{\omega R_2 C} \right) = -135°$$

Even though the source was a sine function, the trig identity provides the cosine, so we use that. Also, the angle correct is tricky — we must be careful to get into the correct quadrant. (With the sine negative and the cosine positive, the angle is in the third quadrant.)