Laplace transforms

When we evaluate the performance of a circuit, there are many aspects to consider. Two of the most important are:

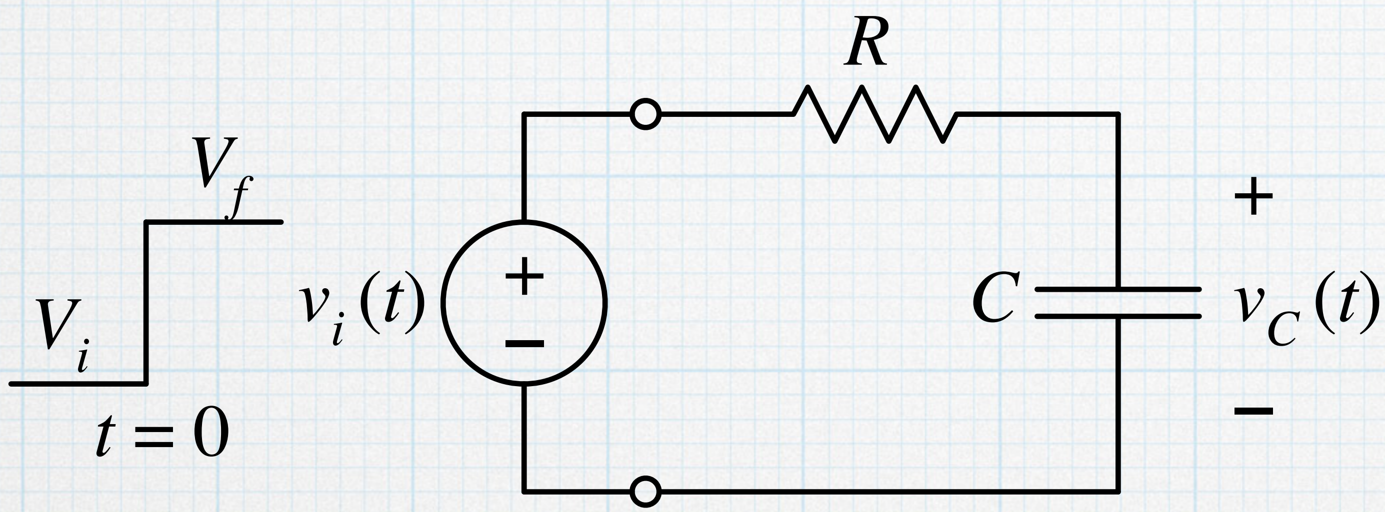
1. Step response. How does the output respond when the input changes abruptly, as in the case of a digital logic circuit? In other words, what is the transient response to large change in input voltage or current?

2. Frequency response. What is the the response at the output when the input is a sinusoid? In particular, how does the sinusoidal output change when the frequency is varied?

Both types of analysis were introduced in EE 201 — *RC*, *RL*, and *RLC* transients and sinusoidal steady-state analysis. The mathematical approach in each case started with a consideration of the differential equations that characterized the circuits, but the two approaches seemed to diverge. The transient analysis followed directly along the differential-equation route, but the AC analysis veered towards using complex numbers, with the circuit being transformed into a new version that was analyzed using complex math.

Recall from 201

The step response of the capacitor voltage in a simple *RC* circuit. The result is a exponential transient with an *RC* time constant. The *RL* transient case had similar behavior.



at $t = 0, v_C = V_i$

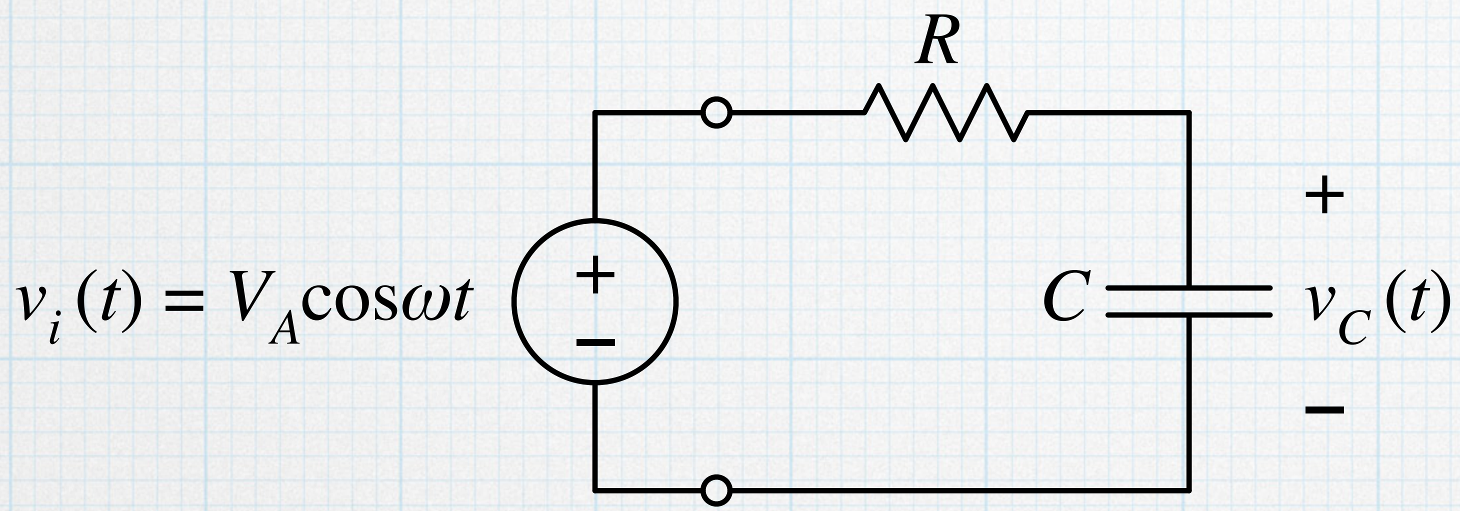
$$\frac{dv_C}{dt} + \frac{v_C}{RC} = \frac{V_f}{RC}$$

$$v_C(t) = V_f - \left[V_f - V_i\right] \exp(-\sigma t) \qquad \sigma = \frac{1}{\tau} = \frac{1}{RC}$$

(Go back the 201 notes, if you need to review.)

Also from 201

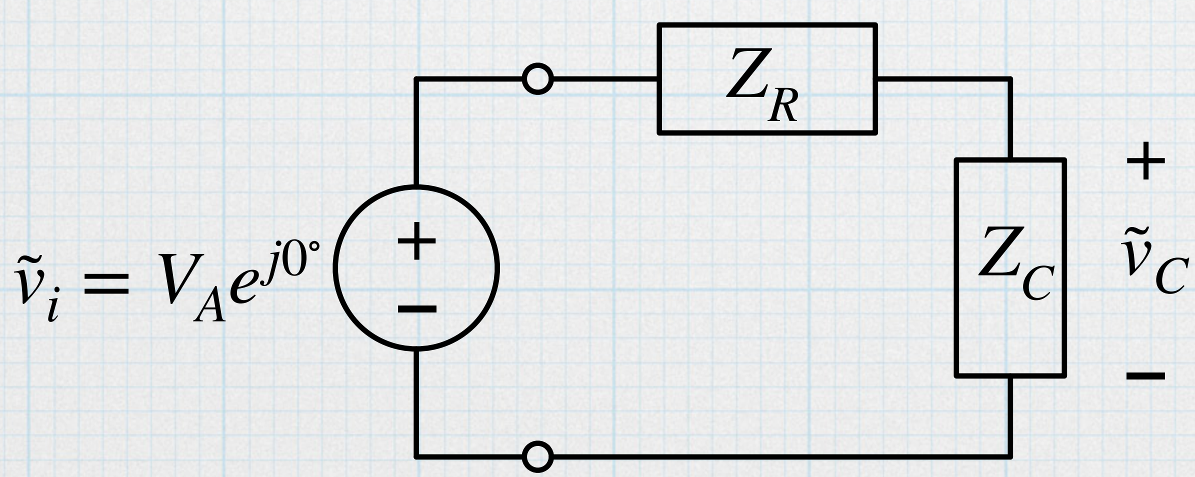
The sinusoidal response of the capacitor voltage in a simple RC circuit. In 201, we solved two ways: as a straight-forward differential equation and then by transforming the circuit and using impedances with complex analysis. With the complex approach, transient effects were ignored.



$$\frac{dv_C}{dt} + \frac{v_C}{RC} = \frac{V_A}{RC}\cos\omega t$$

$$v_C(t) = Ae^{-\sigma t} + M\cos(\omega t + \theta)$$

$$M = \frac{V_A}{\sqrt{1 + (\omega RC)^2}} \qquad \theta = -\arctan(\omega RC)$$

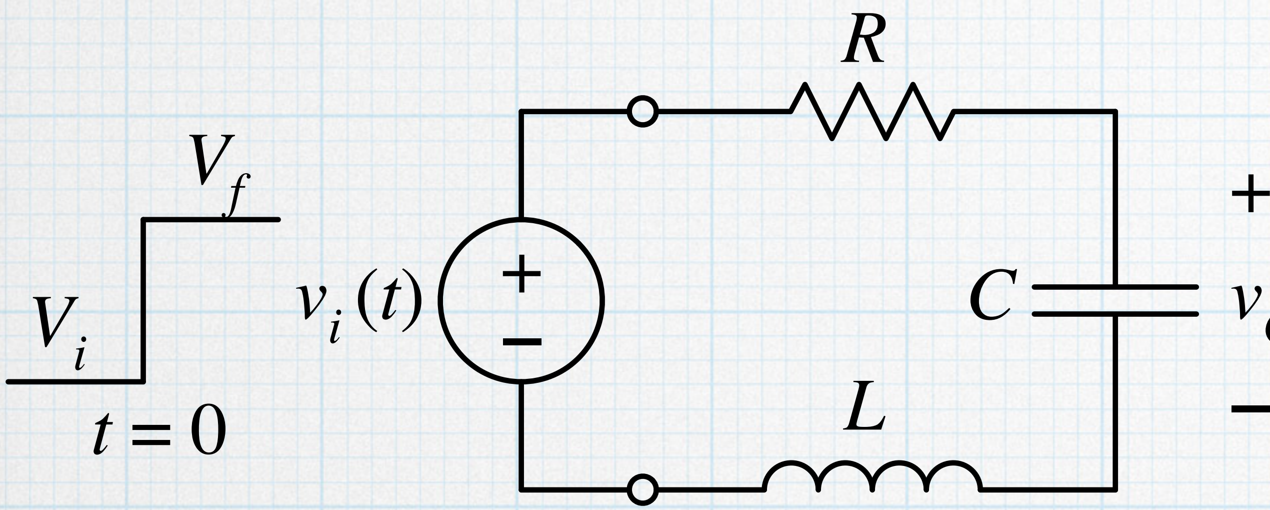


$$\tilde{v}_C = \frac{Z_C}{Z_R + Z_C}\tilde{v}_i$$

$$\tilde{v}_C = Me^{j\theta}$$

$$v_C(t) = M\exp\left(j\omega t + \theta\right)$$

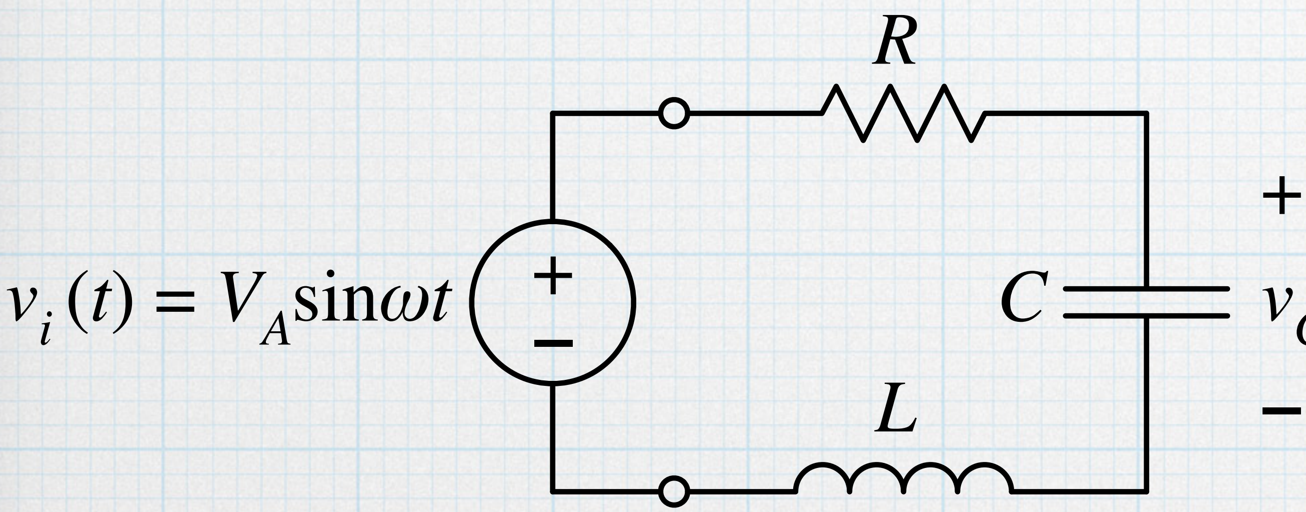
We saw a similar relationships for second-order *RLC* circuits.



Overdamped → two decaying exponentials.

$$v_C(t) = V_f - \left(V_f - V_i\right)\left[A_1 e^{-\sigma_1 t} + A_2 e^{-\sigma_2 t}\right]$$

A_1, A_2, σ_1, and σ_2 depend on R, L, C, and initial conditions.



$$v_C(t) = B_1 e^{-\sigma_1 t} + B_2 e^{-\sigma_2 t} + M\cos(\omega t + \theta)$$

B_1, $B_2, \sigma_1, \sigma_2, M, \theta$ depend on V_m, R, L, C, ω, and initial conditions.

Again, using sinusoidal steady-state analysis, the sinusoidal part of the capacitor voltage can be expressed as $v_C(t) = Me^{j(\omega t + \theta)} \rightarrow \mathbf{v_C} = Me^{j\theta}$

Can these two approaches — the step-response using differential equations and the AC method using a circuit transformation and complex analysis — be reconciled? In both cases, we are looking atet same circuit — only the details of the source have changed. It seems that there might be a more unified approach to handling the two situations. Solving differential equations is tedious but does works in every case. The circuit transform approach offers simplicity, but can it be made more general? The key the unification comes in considering the time dependence of the solutions in the two cases. In particular:

For step-function problems, the solutions were exponentials, characterized by a decay rate (or rates for the 2nd-order case):

$v_C(t) \propto \exp(-\sigma t)$ $\sigma \rightarrow$ decay rate, units of (seconds)$^{-1}$.
(Or nepers/s. Nepers are dimensionless.)

For the steady-state sinusoidal problems, using the complex form, the solutions are also exponentials, but this time complex:

$v_C(t) \propto \exp(j\omega t)$ $\omega \rightarrow$ angular frequency, units of (seconds)$^{-1}$.
(Or rad/s. Radians are dimensionless.)

Complex frequency

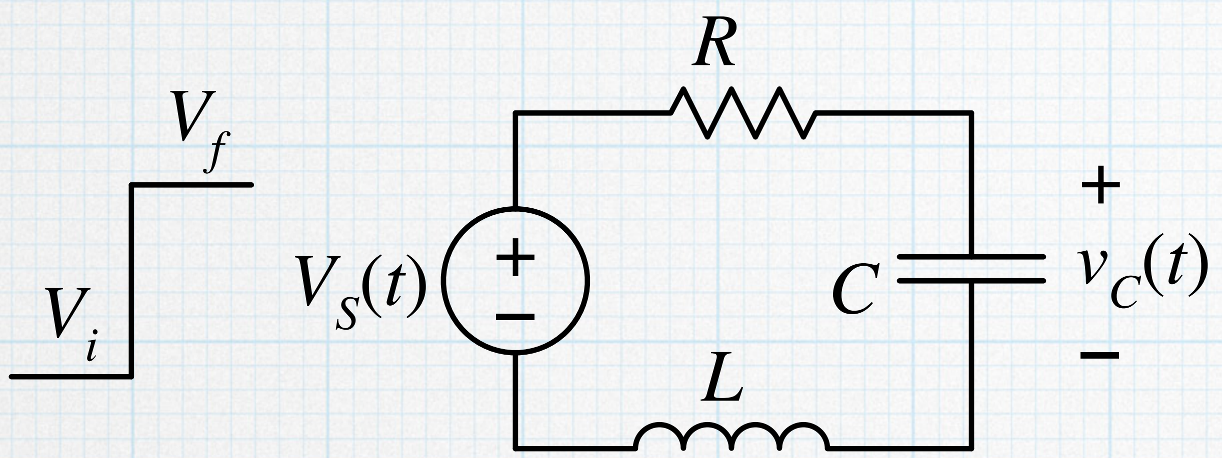
We might consider combining these two exponentials into a single quantity, which we could call *complex frequency*:

$$s = \sigma + j\omega \rightarrow e^{st} = e^{\sigma t} e^{j\omega t} \qquad s \rightarrow \text{complex frequency, units of s}^{-1}.$$

The complex frequency encompasses both transient and sinusoidal situations. For pure step-function situations, $\omega = 0$ (DC) and $s = \sigma$. For sinusoidal steady-state situations, $\sigma = 0$ and $s = j\omega$.

In fact, we witnessed this unified frequency notion in 201. In the underdamped *RLC* transient, the capacitor voltage oscillated for a time before settling to the final voltage — both σ and ω were needed in the solution.

Recall



underdamped → a decaying oscillation

From 201: $$v_C(t) = V_f - \left(V_f - V_i\right) e^{-\sigma t} \left[\cos \omega_d t + \frac{\sigma}{\omega_d} \sin \omega_d t\right]$$

Using this new notion of complex frequency, we can re-write the underdamped response as:

$$v_C(t) = V_f - \left(\frac{V_f - V_i}{2}\right) \left[\left(1 + \frac{\sigma}{\omega_d}\right) e^{-(\sigma - j\omega_d)t} + \left(1 - \frac{\sigma}{\omega_d}\right) e^{-(\sigma + j\omega_d)t}\right]$$

The Laplace Transform

The idea of complex frequency leads inexorably to the Laplace transform which is one of a number of integral transforms that allow for easier solution of differential equations. The idea is to transform a problem from one domain (or space) into a related domain, where, hopefully, the equations are easier to solve. Applying this method to circuits, we will transform the differential equation from the time domain to the frequency domain and find a solution in that form. Then we can transform back to the time domain to arrive at the final solution. You likely saw this method applied in your differential equations class.

However, we will learn soon enough that transforming back from the frequency domain is not really necessary. The frequency-domain representation presents useful information without the need to transform back to the time domain.

Working in the frequency domain is a key skill for EEs. Being able to see how a system behaves in both the time domain and the frequency domain leads to a much deeper understanding of a system and is essential for system design.

The Laplace Transform

Given a function of time, $f(t)$, we can transform it into a new, but related, function $F(s)$.

$$\mathscr{L}\left\{f(t)\right\} = \int_0^\infty f(t)\, e^{-st} dt = F(s)$$

- $\exp(-st)$ is the *kernel* of the transform, where $s = \sigma + j\omega$ is the complex frequency.
- By integrating from 0 to infinity, we "integrate out the time", leaving a function that depends only on s.
- The two variables s and t are *complementary*. If t is time, then s must have units of inverse time, i.e. a frequency, and the product $s{\cdot}t$ is then dimensionless.
- This is the "one-sided" Laplace transform, since the integral starts at $t = 0$. There is a two-sided Laplace transform, but the extra integration range doesn't really add to the utility of the transformation. In using the one-sided version, we assume that everything starts at $t = 0$.
- The variable s is complex, and so $F(s)$ must be complex function. This has implications when we attempt to use $F(s)$ later.

Useful properties of Laplace Transforms

Given functions $f(t)$ and $f_1(t)$, having L.T.s $F(s)$ and $F_1(s)$

$$F(s) = \mathscr{L}\{f(t)\} \qquad F_1(s) = \mathscr{L}\{f_1(t)\}$$

Here are a couple of obvious ones. These relations are easily proved using the definition of at the Laplace transform. We will use these results constantly when applying Laplace transforms to circuits.

1. Multiply / divide by a constant m. (The number m could be complex.)

$$\mathscr{L}\{m \cdot f(t)\} = m \cdot \mathscr{L}\{f(t)\} = m \cdot F(s)$$

$$\mathscr{L}\left\{\frac{f(t)}{m}\right\} = \frac{\mathscr{L}\left\{f(t)\right\}}{m} = \frac{F(s)}{m}$$

2. Addition and subtraction.

$$\mathscr{L}\left\{f(t) \pm f_1(t)\right\} = F(s) \pm F_1(s)$$

Here are the two key relationships for Laplace transforms. Without these, the Laplace method would not be very useful.

3. Differentiation. $f(0)$ is the initial condition of the function at $t = 0$.

$$\mathscr{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0)$$

4. Integration:

$$\mathscr{L}\left\{\int_0^\infty f(t)\,dt\right\} = \frac{F(s)}{s}$$

Higher order derivatives and integrals.

$$\mathscr{L}\left\{\frac{d^2f(t)}{dt^2}\right\} = s^2F(s) - sf(0) - \left.\frac{df}{dt}\right|_{t=0}$$

$$\mathscr{L}\left\{\int_0^\infty \int_0^t f(x)\,dx\right\} = \frac{F(s)}{s^2}$$

Some other interesting properties. (These will be used more extensively in EE 324.)

5. Changing time scale: $\mathscr{L}\left\{f(at)\right\} = \frac{1}{a}F\left(\frac{s}{a}\right)$

Expanding the time scale compresses the frequency scale.
Compressing the time scale expands the frequency scale.

6. Time shift: $\mathscr{L}\left\{f(t-a)\right\} = e^{-as} \cdot F(s)$

7. Frequency shift: $\mathscr{L}\left\{e^{at} \cdot f(t)\right\} = F(s-a)$

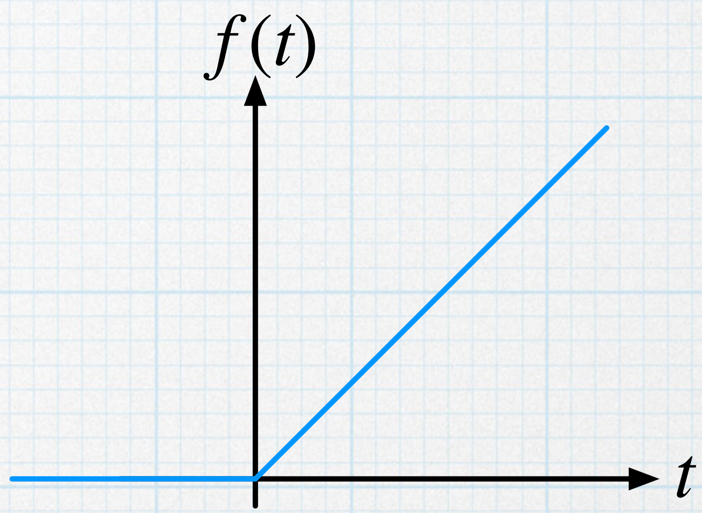
Note the mathematical symmetry of the time and frequency shift relationships.

There are many other important properties of Laplace transforms, but we will leave the more advance details to EE 224, EE 324, and other general systems classes. Here we focus on the essentials needed to understand how our basic electronic systems behave.

Example

For a first example consider a linear ramp in time. The function is zero for $t < 0$, and then ramps up with a slope of 1: $f(t) = t$.

$$F(s) = \mathscr{L}\left\{f(t)\right\} = \int_0^\infty t e^{-st} dt$$



Use integration by parts:

$$\int_0^\infty t e^{-st} dt = -\frac{t}{s} e^{-st} \Big|_0^\infty + \frac{1}{s} \int_0^\infty e^{-st} dt$$

$$= -\frac{t}{s} e^{-st} \Big|_0^\infty - \frac{1}{s^2} e^{-st} \Big|_0^\infty$$

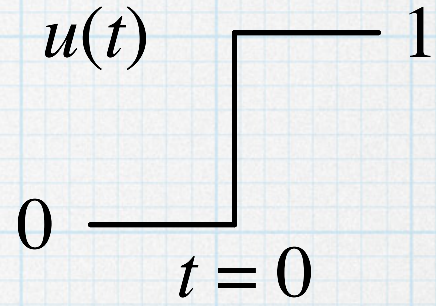
$$= 0 + \frac{1}{s^2}$$

$$F(s) = \frac{1}{s^2}$$

If the slope is not 1, so that $f(t) = m \cdot t$, then $F(s) = \dfrac{m}{s^2}$, by multiplicative property of LTs.

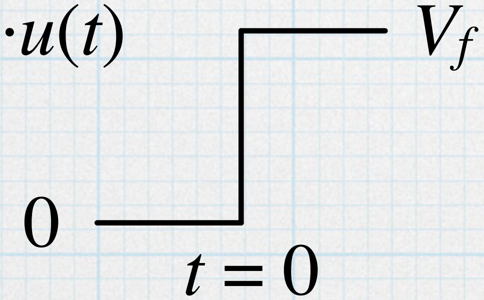
Unit step function

When we studied transients in 201, we frequently used step-change sources, where the source value abruptly jumped from one level to another. However, we didn't develop a mathematical formalism — it was just a jump between two values.

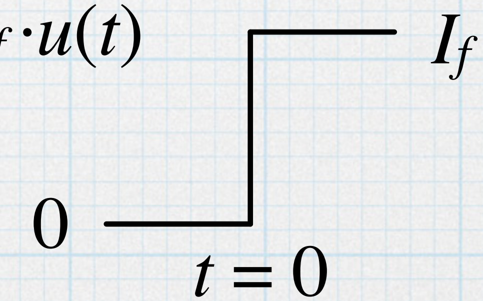


Now we should be more formal. The basic step function, $u(t)$ is defined by an abrupt change from 0 to 1 at $t = 0$, making it a unit step function. Then an abrupt change in source voltage or current can be written as:

$v_s(t) = V_f \cdot u(t)$



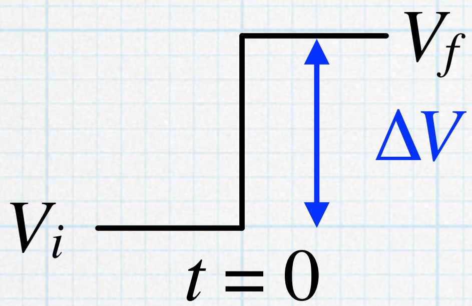
$i_s(t) = I_f \cdot u(t)$



The step could also go negative, in which case a the voltage would be

$$v_s(t) = -V_f \cdot u(t).$$

In 201, we often used the situation where the source started at a non-zero level and stepped to another value.



$$v_s(t) = V_i + \Delta V \cdot u(t)$$

$$= V_f \cdot u(t) + [1 - u(t)]\, V_i$$

If the step occurs at a time $t_o \neq 0$, the step function would be shifted in time, $u\left(t - t_o\right)$.

The unity-step function can also "turn on" other functions so that they are zero for t < 0. For example:

$$v_s(t) = \left[V_A \cos \omega t\right] \cdot u(t)$$

is a function that is 0 for $t < 0$, and then become a cosine for $t \geq 0$.

Since we will be using one-sided Laplace transforms, which are defined from t starting at zero and extending to infinity, we implicitly assume that all source functions are multiplied by a unit step function, so that the functions are definitely zero for $t < 0$.

Transform of unit step, *u(t)*

Apparently, we will need the Laplace transform for the unit step.

$$F(s) = \mathscr{L}\{u(t)\} = \int_0^\infty 1 \cdot e^{-st} dt$$

$$\int_0^\infty e^{-st} dt = -\frac{e^{-st}}{s}\Big|_0^\infty$$

$$= 0 - \left(-\frac{1}{s}\right)$$

$$F(s) = \frac{1}{s}$$

The transform for a typical step voltage source would be:

$$\mathscr{L}\left\{V_a \cdot u(t)\right\} = \frac{V_a}{s}$$

Decaying exponential

Another function that we will use frequently is the simple exponential. We include a decay constant σ.

$$f(t) = e^{-\sigma t}$$

$$\begin{aligned}
\mathscr{L}\left\{e^{-\sigma t}\right\} &= \int_0^\infty e^{-\sigma t} e^{-st} dt \\
&= \int_0^\infty e^{-(s+\sigma)t} dt \\
&= -\frac{1}{s+\sigma} e^{-(s+\sigma)t} \Big|_0^\infty \\
&= \frac{1}{s+\sigma}
\end{aligned}$$

The transform for a growing exponential is

$$\mathscr{L}\left\{e^{+\sigma t}\right\} = \frac{1}{s-\sigma}$$

sinusoids

Recall from Euler: $\cos \omega t = \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)$

$$\mathscr{L}\left\{\cos \omega t\right\} = \frac{1}{2}\int_0^\infty \left(e^{j\omega t} + e^{-j\omega t}\right) e^{-st} dt$$

$$= \frac{1}{2}\int_0^\infty e^{-(s-j\omega)t} dt + \frac{1}{2}\int_0^\infty e^{-(s+j\omega)t} dt$$

$$= -\frac{1}{2(s-j\omega)} e^{-(s-j\omega)t}\Big|_0^\infty - \frac{1}{2(s+j\omega)} e^{-(s-j\omega)t}\Big|_0^\infty$$

$$= \frac{1}{2(s-j\omega)} + \frac{1}{2(s+j\omega)}$$

$$= \frac{s}{s^2+\omega^2}$$

The derivation for sine is similar. The result is:

$$\mathscr{L}\left\{\sin \omega t\right\} = \frac{\omega}{s^2+\omega^2}$$

A few transforms

	$f(t)$	$F(s)$
impulse	$\delta(t)$	1
step	$u(t)$	$\frac{1}{s}$
ramp	t	$\frac{1}{s^2}$
exponential	$e^{-\sigma t}$	$\frac{1}{s+\sigma}$
sine	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
cosine	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
phasor	$e^{j\omega t}$	$\frac{1}{s-j\omega}$
damped sine	$e^{-\sigma t} \sin \omega t$	$\frac{\omega}{(s+\sigma)^2+\omega^2}$
damped cosine	$e^{-\sigma t} \cos \omega t$	$\frac{(s+\sigma)}{(s+\sigma)^2+\omega^2}$

Analyzing a circuit in the frequency domain

Now we are ready to apply the Laplace method to solve a problem. We could start with a generic differential equation, like was done in the differential equations class from the math department. However, we may as well go directly to a circuit, since analyzing circuits in the time domain leads to differential equations.

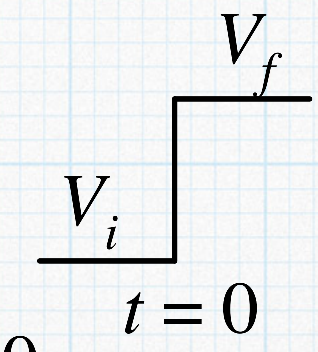
The method is straight-forward:

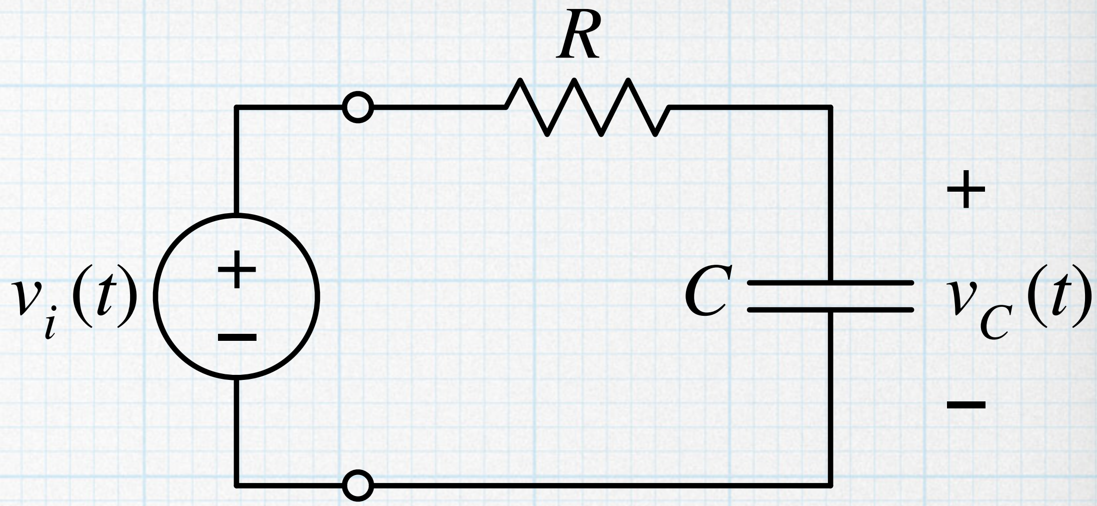
1. Using usual analysis techniques, find the differential equation for the quantity of interest in the circuit.
2. Use Laplace methods to transform the entire equation into the frequency domain. The differential equation in the time domain becomes an algebra problem in the frequency domain.
3. Use basic algebra to find a frequency-domain expression for the Laplace transform of the quantity of interest.
4. Transform back from the frequency domain to the time domain.

Later, we will see that step 4 is optional — it is not always necessary to transform back. We will not to do step 4 in the following examples. We will look the inverse transformation in the next set of notes.

Example 1

Find the frequency domain expression for the capacitor voltage in the *RC* circuit at right.





The source is a unit step with $V_i = 0$ for $t < 0$ and $V_f = 10$ V for $t \geq 0$.

The differential equation has been derived previously, but we will repeat here, just to be complete in this first example.

$$i_R = i_C$$

$$\frac{v_i(t) - v_C(t)}{R} = C\frac{dv_C(t)}{dt}$$

$$\frac{dv_C(t)}{dt} + \frac{v_C(t)}{RC} = \frac{v_i(t)}{RC}$$

Take the the Laplace transform of both sides of the equation

$$\mathscr{L}\left\{ \frac{dv_C(t)}{dt} + \frac{v_C(t)}{RC} \right\} = \mathscr{L}\left\{ \frac{v_i(t)}{RC} \right\}$$

$$\mathscr{L}\left\{\frac{dv_C(t)}{dt}+\frac{v_C(t)}{RC}\right\}=\mathscr{L}\left\{\frac{v_i(t)}{RC}\right\}$$

Use the addition/multiplication properties of the LT to break it down a bit.

$$\mathscr{L}\left\{\frac{dv_C(t)}{dt}\right\}+\frac{1}{RC}\mathscr{L}\left\{v_C(t)\right\}=\frac{1}{RC}\mathscr{L}\left\{v_i(t)\right\}$$

The three transforms are:

$$\mathscr{L}\left\{v_C(t)\right\}=V_C(s)$$

$$\mathscr{L}\left\{\frac{dv_C(t)}{dt}\right\}=sV_C(s)-v_C(0)$$

$$\mathscr{L}\left\{v_i(t)\right\}=\mathscr{L}\left\{V_f\cdot u(t)\right\}=\frac{V_f}{s}$$

Note that in this case, the initial condition is $v_c(0)=0$, so the derivative expression is simplified. Putting it all back together,

$$sV_C(s)+\frac{V_C(s)}{RC}=\frac{V_f}{sRC}$$

$$sV_C(s) + \frac{V_C(s)}{RC} = \frac{V_f}{sRC}$$

With a bit of simple algebra, the frequency domain form of the capacitor voltage is

$$V_C(s) = \frac{1}{s\left(s + \frac{1}{RC}\right)} \cdot \frac{V_f}{RC}$$

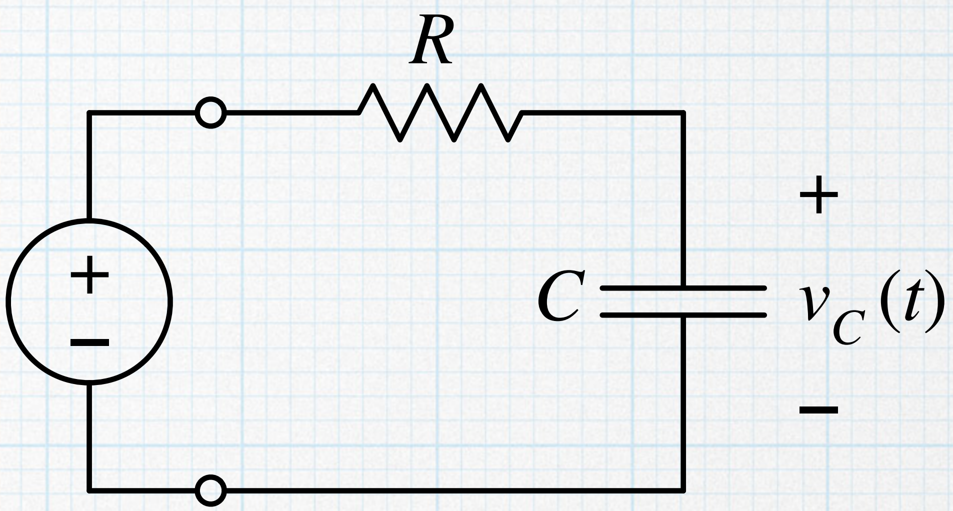
Of course, we don't yet know the meaning of this function. In principle we can transform it back to the time domain, and we will do that soon enough.

More importantly, after a bit more practice, we will come realize that the frequency-domain form above tells us everything we need to know about the circuit's behavior. And arriving at the frequency-domain expression using the Laplace transform was quite easy.

Example 2

Same circuit but with a sinusoid source. Again, assume that $v_c(0) = 0$.

$v_i(t) = V_A \cos\omega t$



We won't go through all the steps here — you should do that on your own. The steps are very similar to example 1.

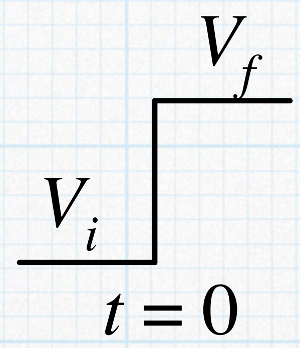
$$\frac{dv_C}{dt} + \frac{v_C}{RC} = \frac{V_A}{RC}\cos\omega t$$

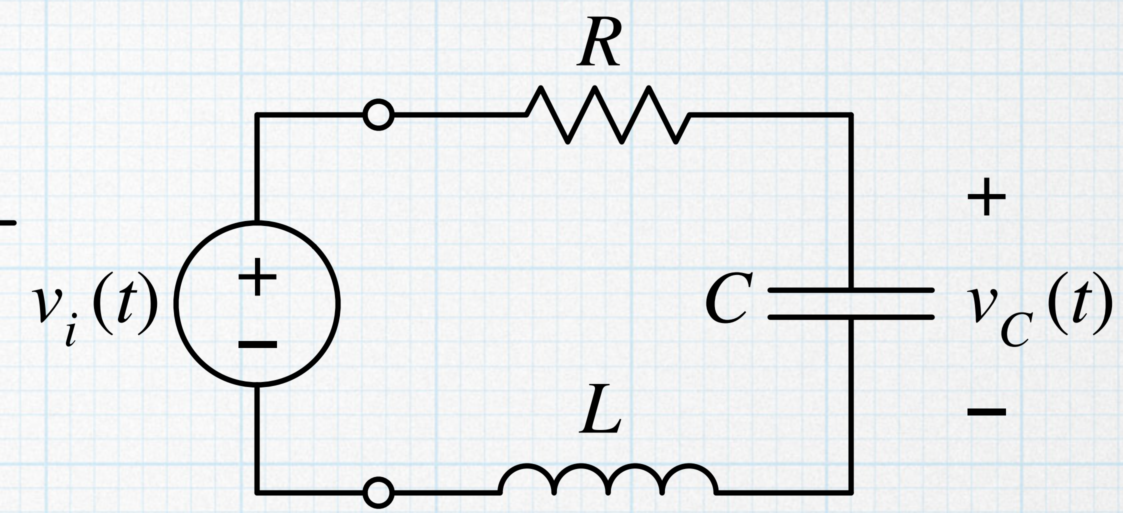
$$sV_C(s) + \frac{V_C(s)}{RC} = \frac{V_A}{RC} \cdot \frac{s}{s^2 + \omega^2}$$

$$V_C(s) = \frac{1}{\left(s^2 + \omega^2\right)\left(s + \frac{1}{RC}\right)} \cdot \frac{V}{RC}$$

Example 3

RLC — a second-order system — with a step input.





Again, the analysis is abbreviated — fill in the missing steps for yourself.

To keep it simple, use initial conditions, $v_C(0) = 0$

and $i_C(0) = 0 \left[\left. \frac{dv_C}{dt} \right|_{t=0} = 0 \right]$.

$$\frac{d^2 v_C(t)}{dt^2} + \frac{R}{L}\frac{dv_C(t)}{dt} + \frac{1}{LC}v_C(t) = \frac{V_f}{LC}u(t)$$

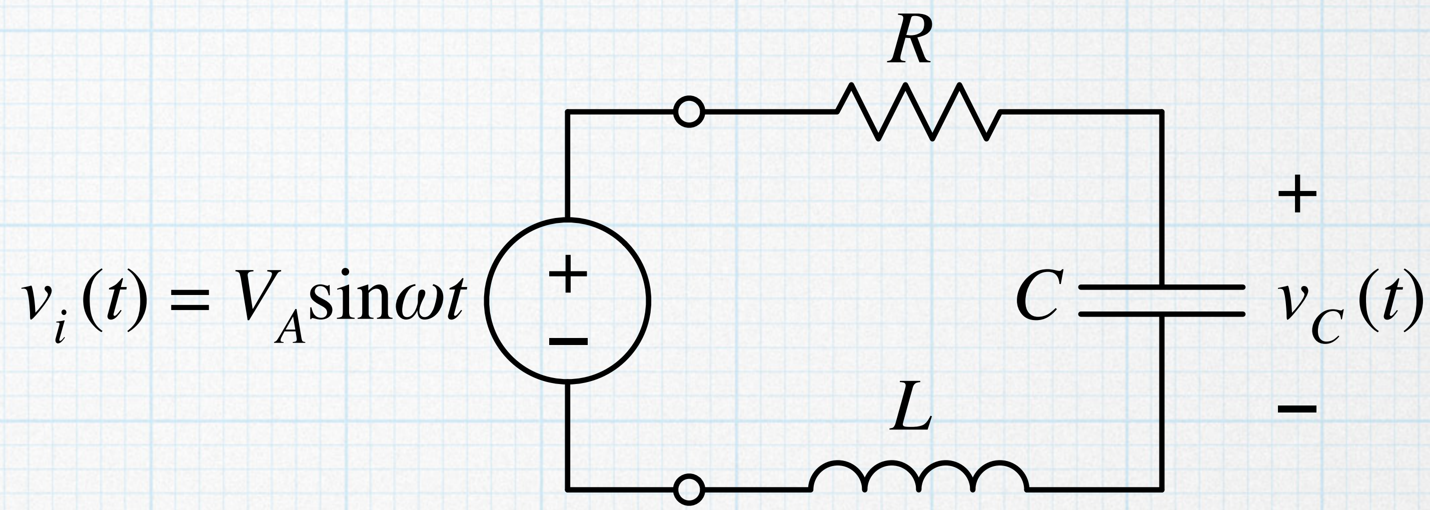
$$s^2 V_C(s) + \frac{R}{L}sV_C(s) + \frac{1}{LC}V_C(s) = \frac{V_f}{sLC}$$

$$V_C(s) = \frac{1}{s\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)} \cdot \frac{V_f}{LC}$$

Once the differential equation is in place, the Laplace stuff is so easy!

Example 4

An *RLC* with a sine function source. Use the same initial conditions as in the previous *RLC* example.



$$\frac{d^2 v_C(t)}{dt^2} + \frac{R}{L}\frac{dv_C(t)}{dt} + \frac{1}{LC}v_C(t) = V_A \sin \omega t$$

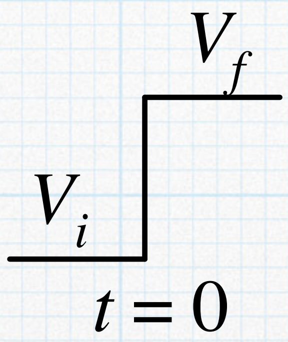
$$s^2 V_C(s) + \frac{R}{L} s V_C(s) + \frac{1}{LC} V_C(s) = \frac{V_A}{LC} \cdot \frac{\omega}{s^2 + \omega^2}$$

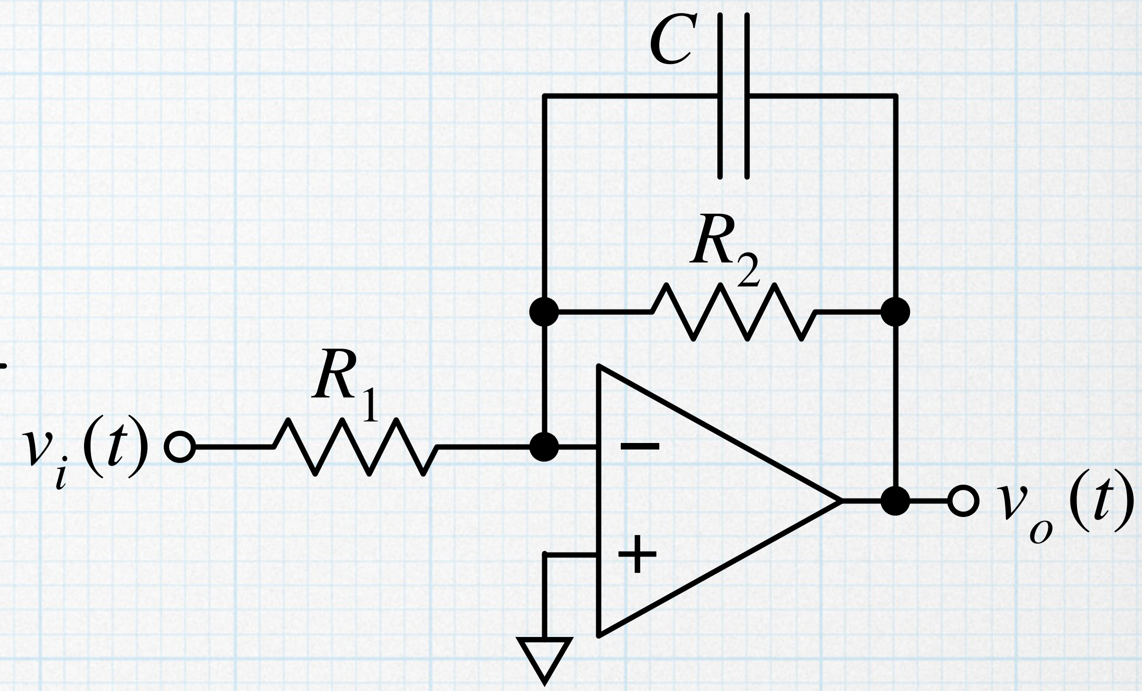
$$V_C(s) = \frac{\omega}{\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)\left(s^2 + \omega^2\right)} \cdot \frac{V_f}{LC}$$

Example 5

How about an op amp with a step input? Again, for simplicity, use initial condition of $V_i = 0$, which translates to

$v_C(0) = 0$ [$v_o(0) = 0$].





$$i_{R1} = i_{R2} + i_C$$

$$\frac{v_i(t)}{R_1} = \frac{V_f \cdot u(t)}{R_1} = \frac{-v_o(t)}{R_2} - C\frac{dv_o}{dt}$$

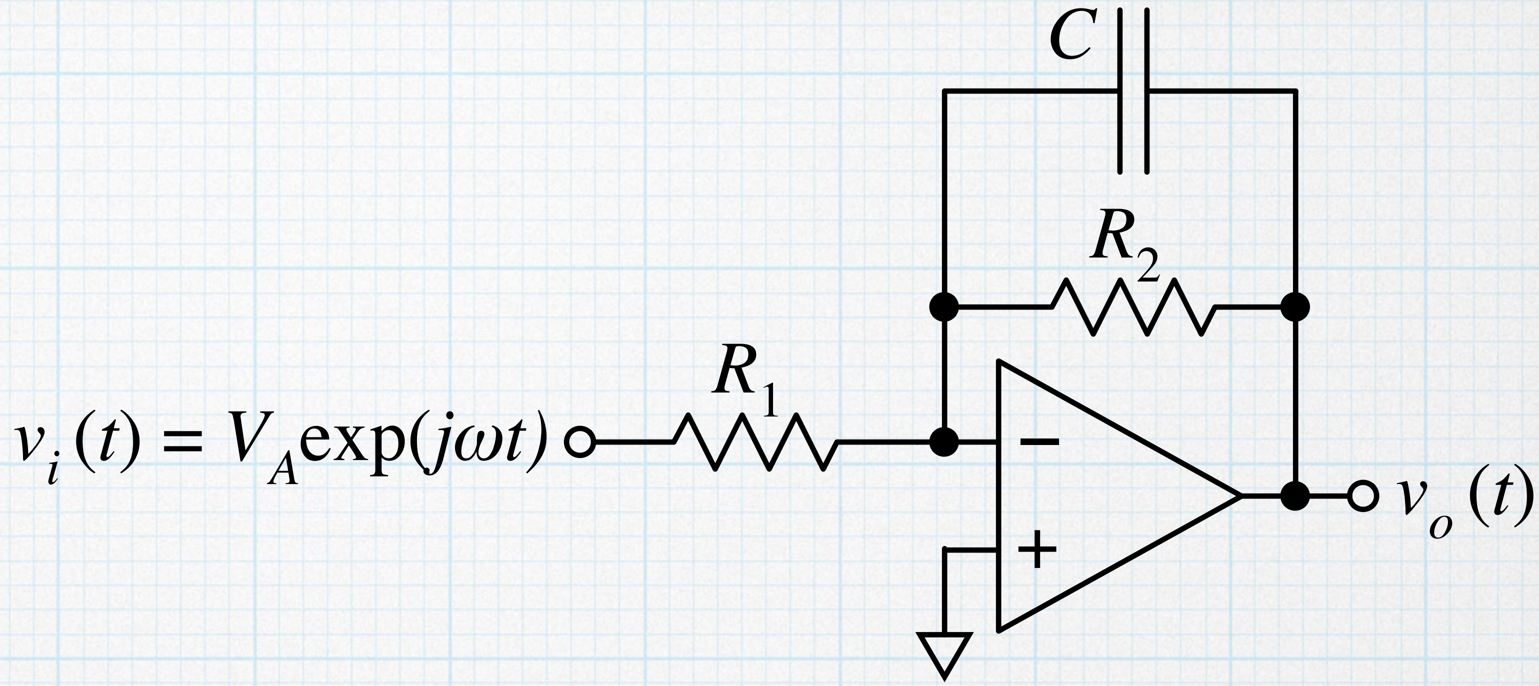
$$\frac{V_f}{sR_1} = -\frac{V_o(s)}{R_2} - CsV_o(s)$$

$$V_o(s) = -\frac{1}{s\left(s + \frac{1}{R_2 C}\right)} \cdot \frac{V_f}{R_1 C}$$

Note how similar this is to the result of Example 1.

Example 6

One more. Let's do an op amp with a sinusoidal source. Just for fun, use a complex exponential for the input sinusoid.



$$\frac{v_i(t)}{R_1} = \frac{V_A \cdot e^{j\omega t}}{R_1} = \frac{-v_o(t)}{R_2} - C\frac{dv_o}{dt}$$

$$\frac{V_f}{R_1\left(s - j\omega\right)} = -\frac{V_o(s)}{R_2} - CsV_o(s)$$

$$V_o(s) = -\frac{1}{\left(s + \frac{1}{R_2 C}\right)\left(s - j\omega\right)} \cdot \frac{V_f}{R_1 C}$$

Units

In each of the examples, once the differential equation was transformed into the frequency domain, the math was quite easy. That is the magic of Laplace transforms — they turn messy differential equations is simpler algebra equations. Again, you may well have seen this your diff. eq. math class.

However, there is an important distinction that we should emphasize. In a typical math class, the variables and equations have no units. When s was introduced as complementary variable in the LT process, it did not have any specific physical significance — it was just a means to an end. Transform a function of t (or x or z or whatever) into a function of s, do some manipulations, and then transform back to the original variable. The "s" fades away.

On the other hand, our circuits are physical, and the every quantity has units that are tied to the physical meaning. So in transforming from time to frequency, the units for s must be complementary — s is the complex frequency and it must have units of inverse seconds (sec^{-1}). (Note the potential confusion if we use s for seconds as is typical in most situations.)

Similarly, the transforms for voltage and current have defined units. In looking at the definition of the LT, it is apparent that voltage in the time domain transforms to a frequency-domain quantity with units of volt-seconds (V·sec.) A frequency-domain current has units of amp-seconds (A·sec). These are subtle points, but they are important.