Solving circuits directly using Laplace

The Laplace method seems to be useful for solving the differential equations that arise with circuits that have capacitors and inductors and sources that vary with time (steps and sinusoids.) The approach has been to:

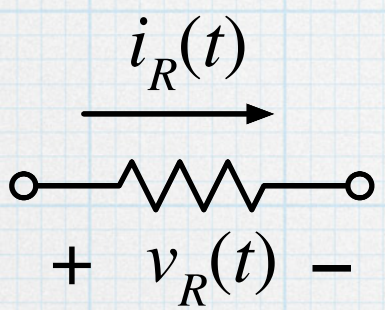
1. Analyze the circuit in the time domain using familiar circuit analysis techniques to arrive at a differential equation for the time-domain quantity of interest (voltage or current).
2. Perform a Laplace transform on the differential equation to arrive a frequency-domain form of the quantity of interest.
3. Solve the frequency-domain algebra expression.
4. Transform back to the time-domain.

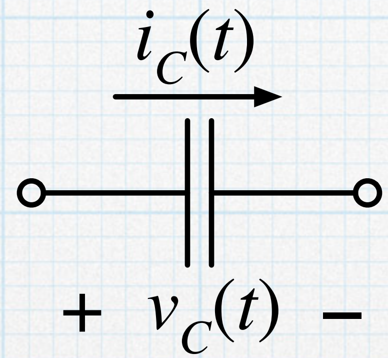
Might it possible to change the order of the steps? Could we transform the circuit into the frequency domain and then use circuit techniques to find the desired voltage or current? Might this is approach be easier than solving differential equations?

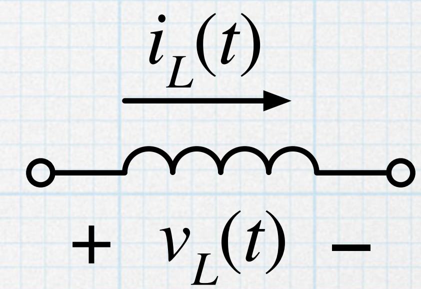
Not surprisingly, the answer to all three questions is "Yes!"

Frequency domain impedances

In order to transform a circuit directly, we need frequency-domain descriptions of the all of the components in the circuit. We already know how to transform the commonly used step and sinusoidal sources. We need to consider resistors, inductors, and capacitors to see the form of the current-voltage relationships in the frequency domain. Apply the Laplace transform to the *i*-*v* equations directly.



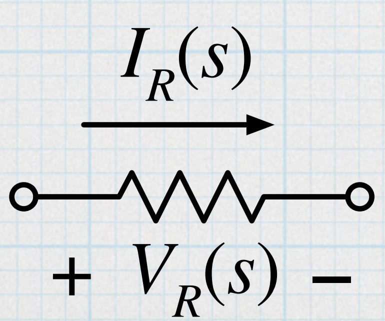


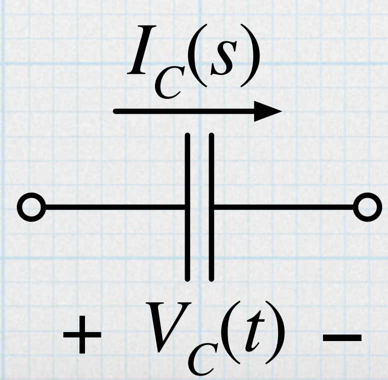


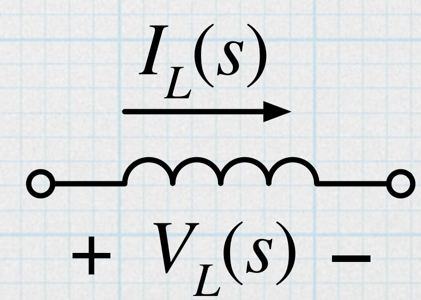
$$v_R(t) = R \cdot i_R(t) \qquad i_C(t) = C\frac{dv_C(t)}{dt} \qquad v_L(t) = L\frac{di_L(t)}{dt}$$

$$V_R(s) = R \cdot I_R(s) \qquad I_C(s) = C \cdot s \cdot V_C(s) \qquad V_L(s) = L \cdot s \cdot I_L(s)$$

$$\frac{V_R}{I_R} = R \qquad \frac{V_C}{I_C} = \frac{1}{sC} \qquad \frac{V_L}{I_L} = sL$$



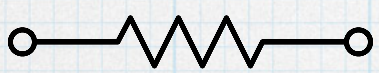




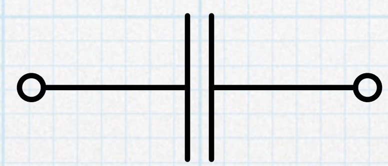
For the resistor, the frequency domain relationship is exactly the same as the the time domain. (Ohm's Law is not time-dependent, so this is not a surprise.) For the inductor and capacitor, the frequency domain relations are actually simpler. All three components can be treated with a simple "Ohm's-Law-like" current-voltage equation:

$$V(s) = Z \cdot I(s)$$

where Z is known as the "impedance", with units of ohms (Ω).



$$Z_R = R$$



$$Z_C = \frac{1}{sC}$$

$$Z_L = sL$$

Z_C and Z_L depend on frequency, but for a given frequency, they are constants. They are complex constants (since s is complex), but the frequency domain relationships are exactly like those of the resistor: voltage is equal to a constant multiplied by the current. This means that the circuit in the frequency domain can be solved using all of the methods that we learned for circuits with sources and resistors at the very beginning of EE 201.

All of the familiar techniques learned in 201 apply in the frequency domain, as well:

- equivalent resistances (now equivalent impedances)
- voltage / current dividers *
- source transformations
- node voltages *
- mesh currents
- superposition

Of course this frequency-domain approach is very similar to the complex analysis used for AC circuits in EE 201. In fact, AC analysis as introduced 201 is simply a special case of the Laplace approach. In our Laplace expressions, if we restrict the complex frequency to just imaginary values, $s = j\omega$, the two approaches become identical.

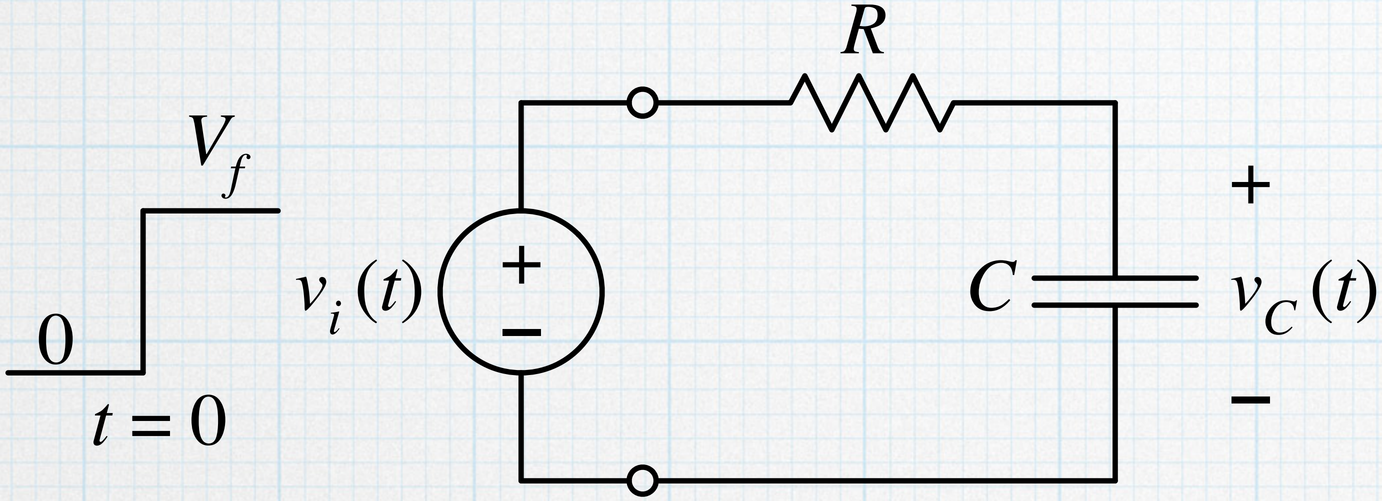
Now, with the approach of transforming the circuit into the frequency domain using impedances, the Laplace procedure becomes:

1. Transform the circuit. Use the Laplace transform version of the sources and the other components become impedances.
2. Solve the circuit using any (or all) of the standard circuit analysis techniques to arrive at the desired voltage or current, expressed in terms of the frequency-domain sources and impedances.
3. Transform back to the time-domain. (If needed.)

The following examples illustrate the method.

Example 1

Find the Laplace (frequency domain) expression for v_C in the RC circuit below. The input is a step function, $v_i(t) = V_f \cdot u(t)$

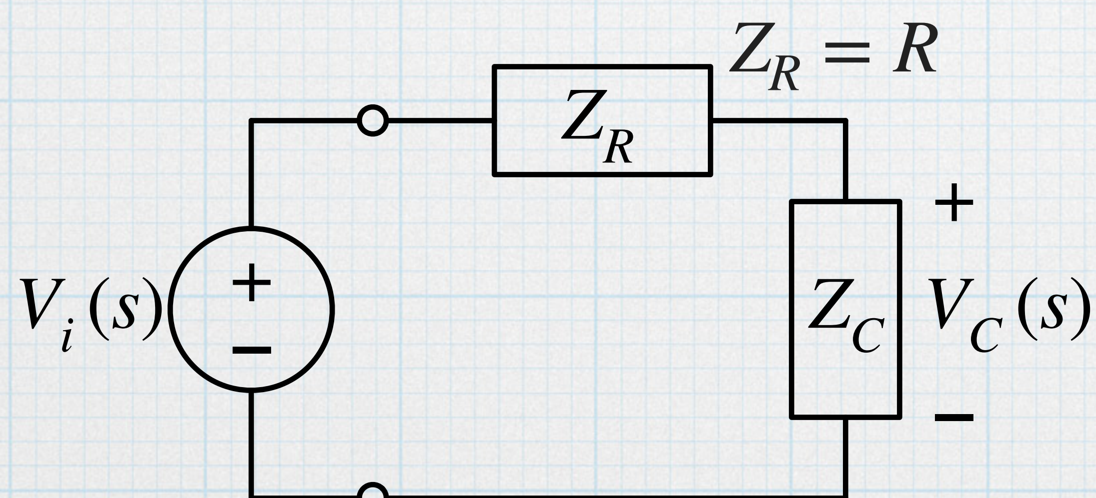


The frequency domain circuit is easily solved using a voltage divider.

$$V_C(s) = \frac{Z_C}{Z_C + Z_R} V_i(s)$$

$$= \left(\frac{\frac{1}{sC}}{\frac{1}{sC} + R} \right) V_i(s)$$

Convert the circuit to the frequency domain.



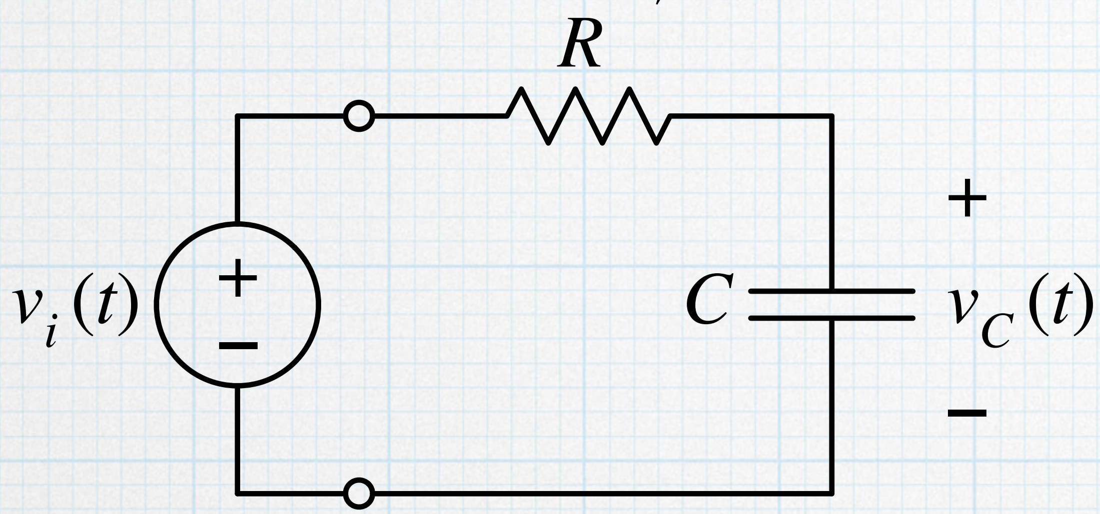
$$V_i(s) = \frac{V_f}{s} \qquad Z_C = \frac{1}{sC}$$

$$V_C(s) = \left(\frac{\frac{1}{RC}}{s + \frac{1}{RC}} \right) \left(\frac{V_f}{s} \right)$$

$$= \frac{V_f/_{RC}}{s\left(s + \frac{1}{RC}\right)}$$

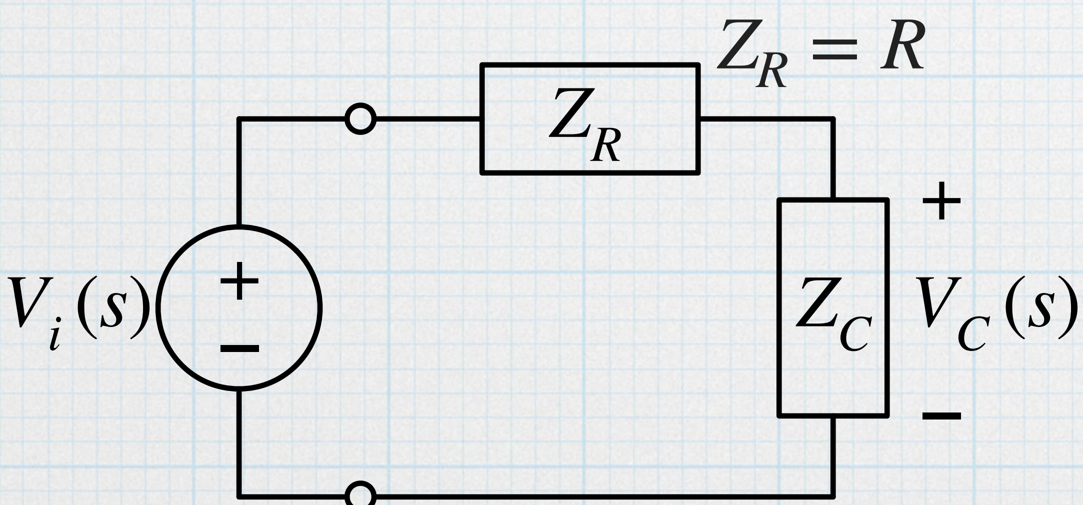
Example 2

The same RC circuit, but now with a sinusoidal source, $v_i = V_A \cos(\omega t)$.



$$v_i(t) = V_A \cdot \cos(\omega t)$$

Convert the circuit to the frequency domain. It looks familiar.



$$V_i(s) = V_A \cdot \frac{s}{s^2 + \omega^2} \qquad Z_C = \frac{1}{sC}$$

The R and C impedances still form a voltage divider.

$$V_C(s) = \frac{Z_C}{Z_C + Z_R} V_i(s)$$

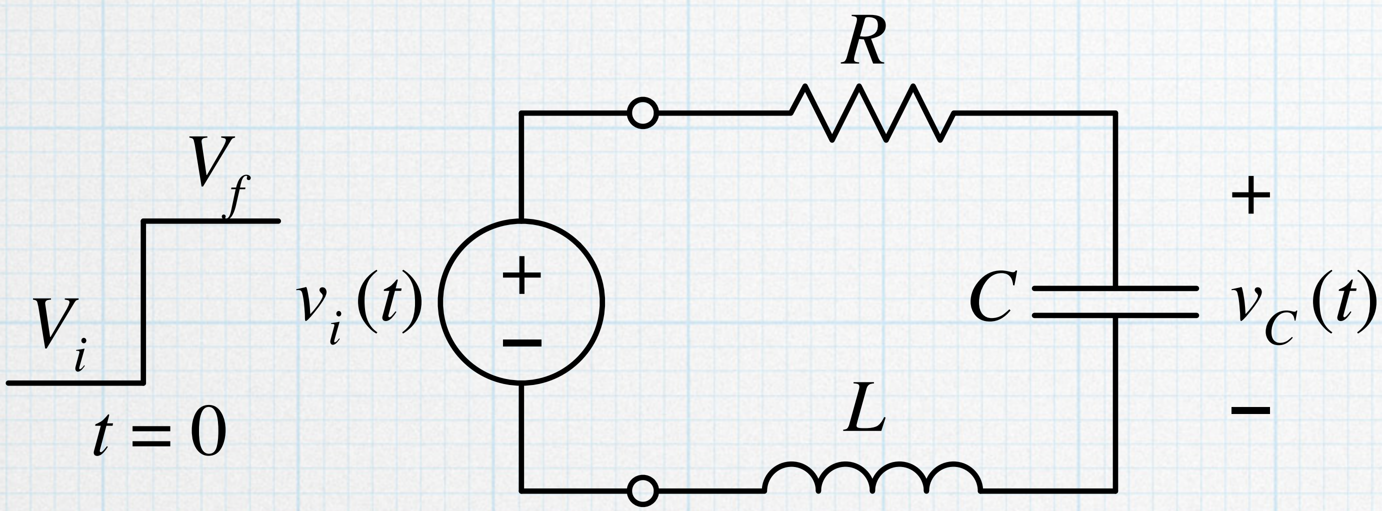
$$= \left(\frac{\frac{1}{sC}}{\frac{1}{sC} + R} \right) V_i(s)$$

$$V_C(s) = \left(\frac{\frac{1}{RC}}{s + \frac{1}{RC}} \right) \left(\frac{V_A \cdot s}{s^2 + \omega^2} \right)$$

$$= \frac{\frac{V_A}{RC} \cdot s}{\left(s + \frac{1}{RC} \right) \left(s^2 + \omega^2 \right)}$$

Example 3

Find the Laplace (frequency domain) expression for v_C in the *RLC* circuit below. The input is a step function, $v_i(t) = V_f \cdot u(t)$.



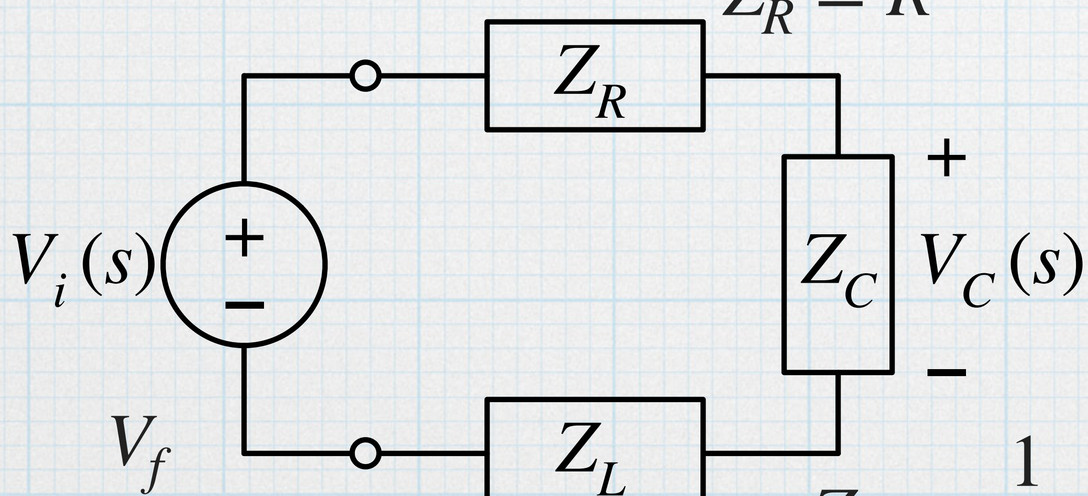
This can still be handled as a voltage divider.

$$V_C(s) = \frac{Z_C}{Z_R + Z_C + Z_L} V_i(s)$$

$$= \left(\frac{\frac{1}{sC}}{R + \frac{1}{sC} + sL} \right) V_i(s)$$

Convert to the frequency domain.

$Z_R = R$



$$V_i(s) = \frac{V_f}{s}$$

$Z_L = sL$

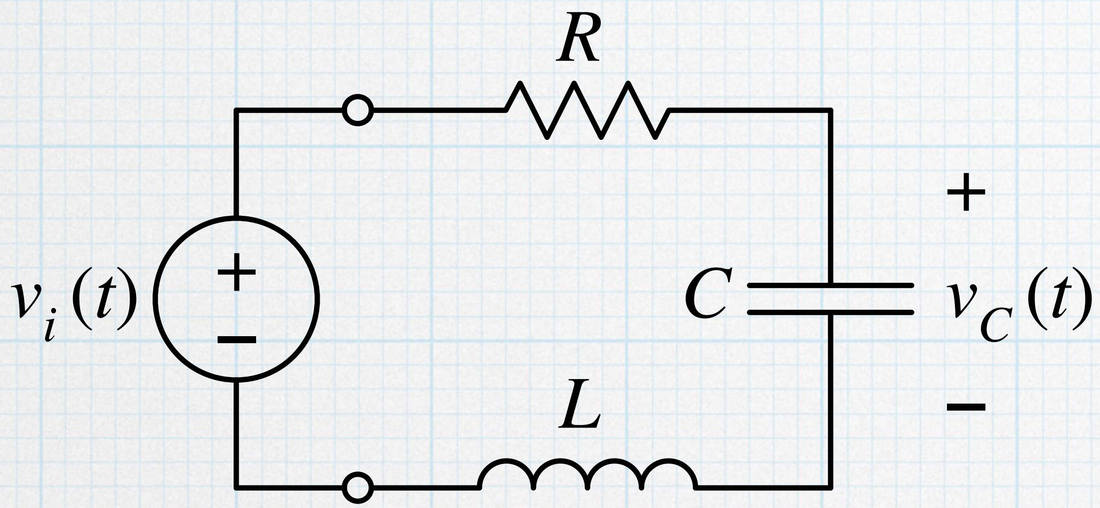
$$Z_C = \frac{1}{sC}$$

$$V_C(s) = \left(\frac{1}{1 + sRC + s^2 LC} \right) \left(\frac{V_f}{s} \right)$$

$$= \frac{\frac{V_f}{LC}}{s\left(s^2 + s\frac{R}{L} + \frac{1}{LC} \right)}$$

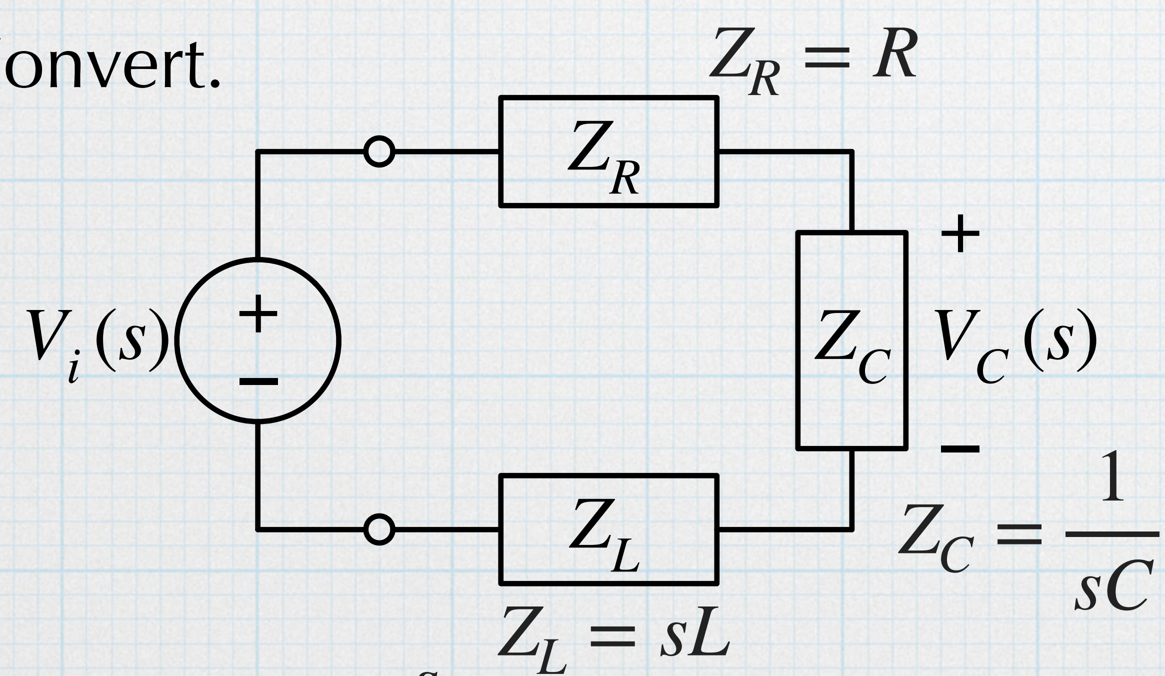
Example 4

The same *RLC* circuit as Example 3, but now with a sinusoidal source, $v_i = V_A \cos(\omega t)$.



$$v_i(t) = V_A \cdot \cos(\omega t)$$

Convert.



$$V_i(s) = V_A \cdot \frac{s}{s^2 + \omega^2}$$

Yep, it's still a voltage divider.

$$V_C(s) = \frac{Z_C}{Z_R + Z_C + Z_L} V_i(s)$$

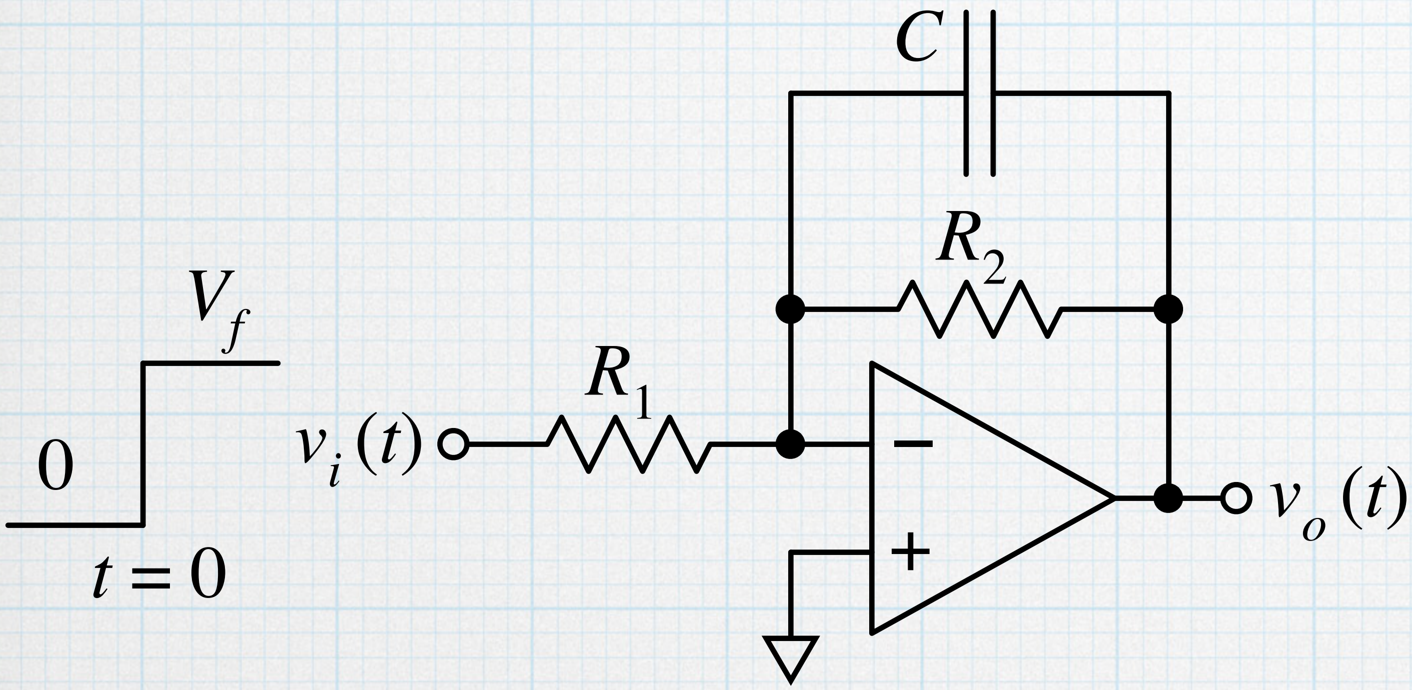
$$= \left(\frac{\frac{1}{sC}}{R + \frac{1}{sC} + sL} \right) V_i(s)$$

$$V_C(s) = \left(\frac{1}{1 + sRC + s^2 LC} \right) \left(\frac{V_A}{s^2 + \omega^2} \right)$$

$$= \frac{\frac{V_A}{LC}}{\left(s^2 + \omega^2\right) \left(s^2 + s\frac{R}{L} + \frac{1}{LC} \right)}$$

Example 5

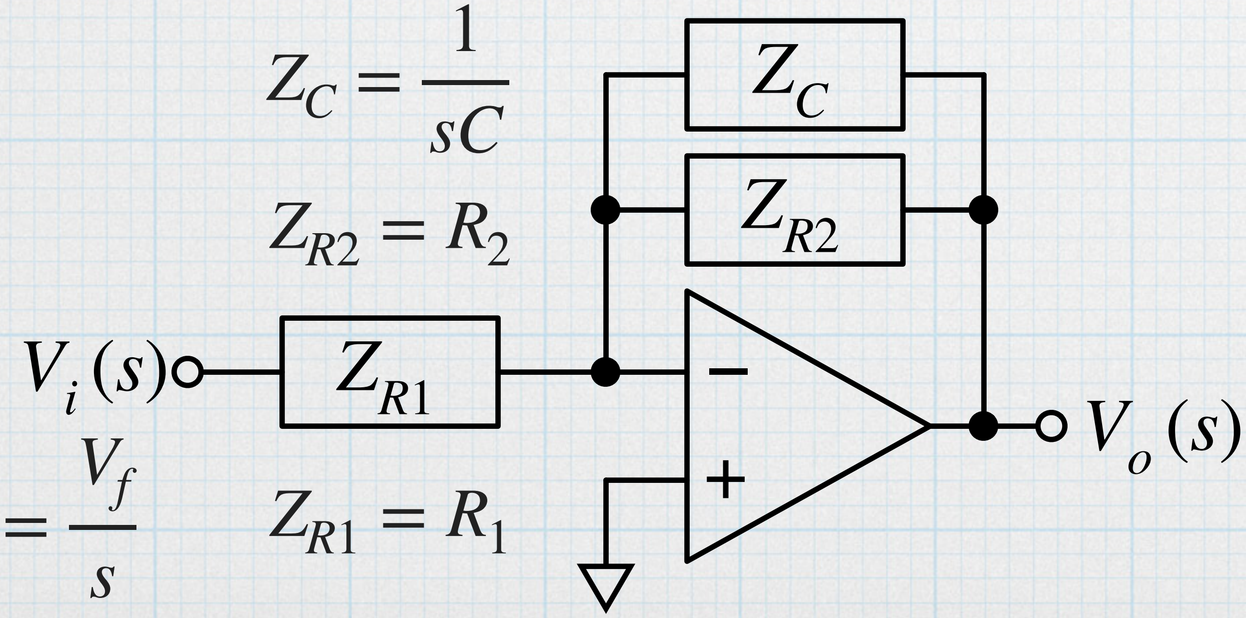
Let's try an op-amp with a step-voltage input. Find the frequency-domain expression for the output, $V_o(s)$.



Convert to the frequency domain.

$$Z_C = \frac{1}{sC}$$

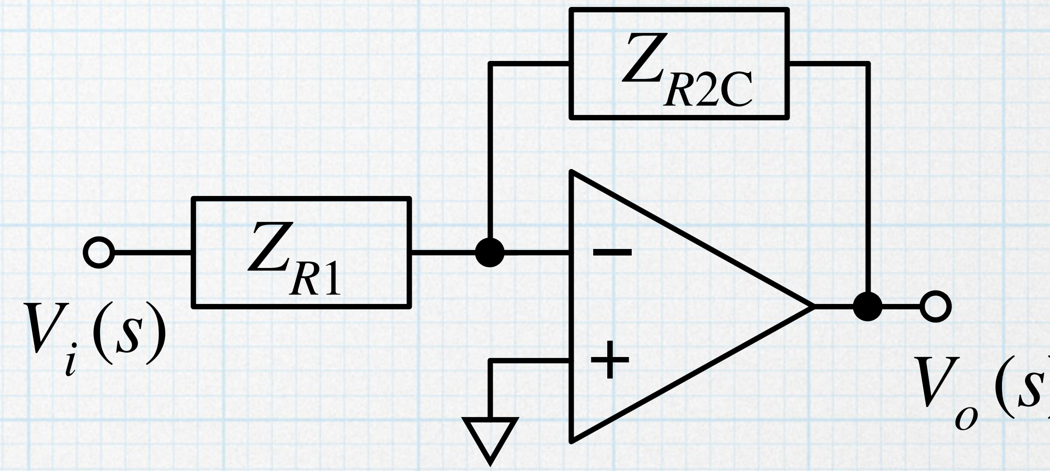
$$Z_{R2} = R_2$$



$$V_i(s) = \frac{V_f}{s} \qquad Z_{R1} = R_1$$

First, combine Z_{R2} and Z_C to make the parallel equivalent.

$$Z_{R2C} = R_2 \| \left(\frac{1}{sC} \right) = \frac{R_2}{1 + sR_2C}$$



It is just a simple inverting amp!

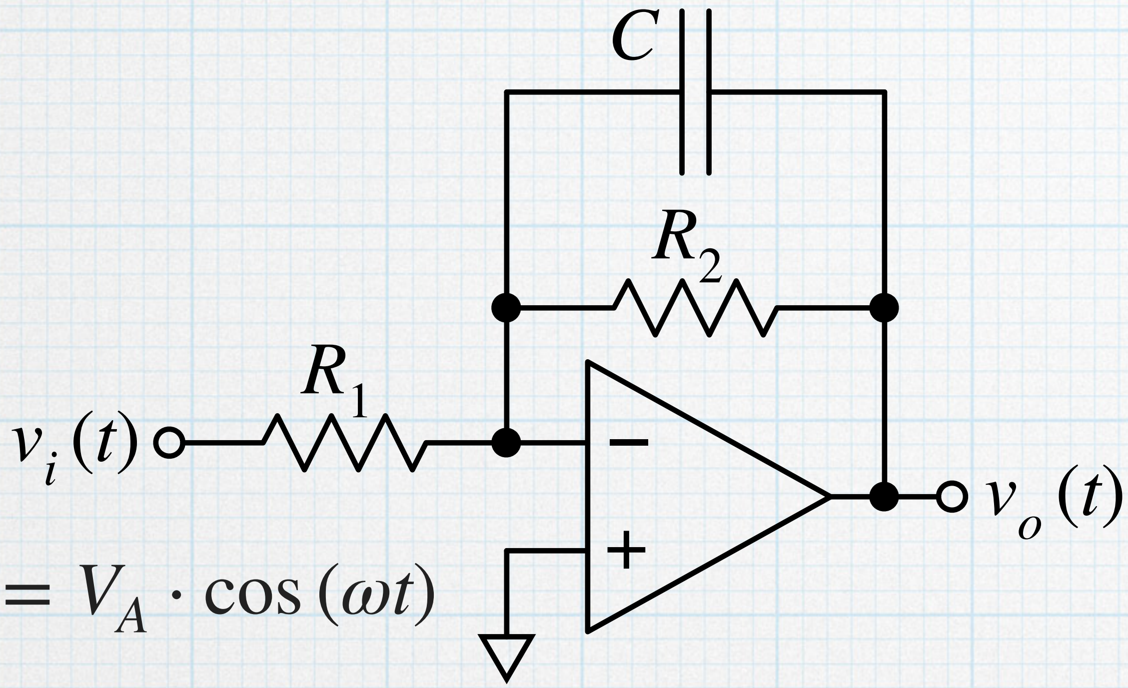
$$V_o(s) = \left(-\frac{Z_{R2C}}{Z_{R1}} \right) V_i(s)$$

$$= \left(-\frac{\frac{R_2}{R_1}}{1 + sR_2C} \right) \left(\frac{V_f}{s} \right)$$

$$= -\frac{\frac{V_f}{R_1C}}{s\left(s + \frac{1}{R_2C} \right)}$$

Example 6

Same op amp circuit, but now with a sinusoidal input.



$$v_i(t) = V_A \cdot \cos(\omega t)$$

Once again, it is just an inverting amp.

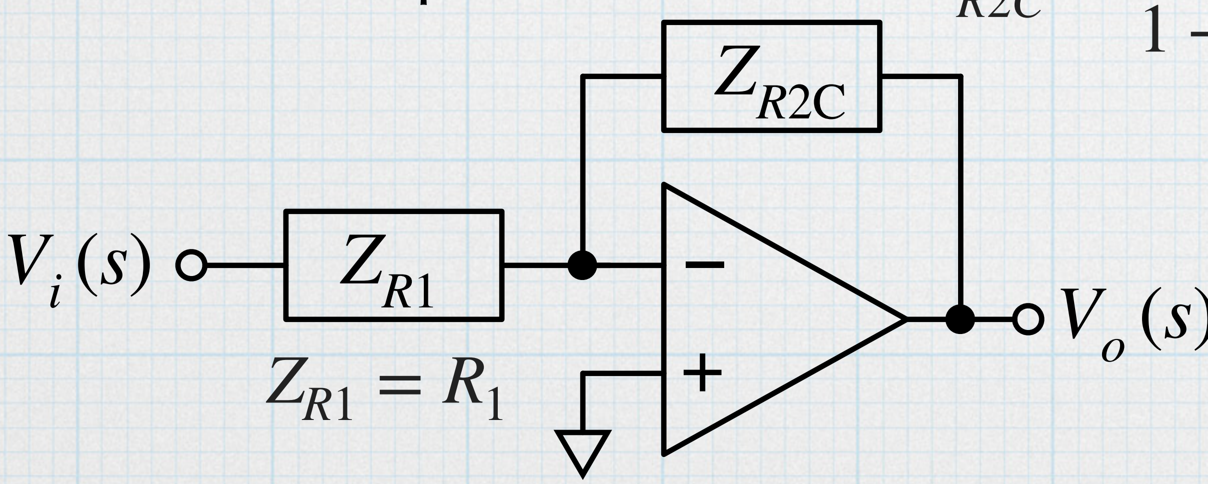
$$V_o(s) = \left(-\frac{Z_{R2C}}{Z_{R1}}\right) V_i(s)$$

$$= \left(-\frac{\frac{R_2}{R_1}}{1 + sR_2C}\right)\left(\frac{V_A \cdot s}{s^2+\omega^2}\right)$$

$$= -\frac{\frac{V_A}{R_1C} \cdot s}{\left(s^2+\omega^2\right)\left(s + \frac{1}{R_2C}\right)}$$

Convert to the frequency domain. Combine Z_{R2} and Z_C as in the previous example.

$$Z_{R2C} = \frac{R_2}{1 + sR_2C}$$



$$Z_{R1} = R_1$$

$$V_i(s) = V_A \cdot \frac{s}{s^2+\omega^2}$$