First-order filters

The general form for the transfer function of a first order filter is:

$$T(s) = \frac{a_1 s + a_0}{b_1 s + b_o}$$

However, we will typically recast this into a standard form:

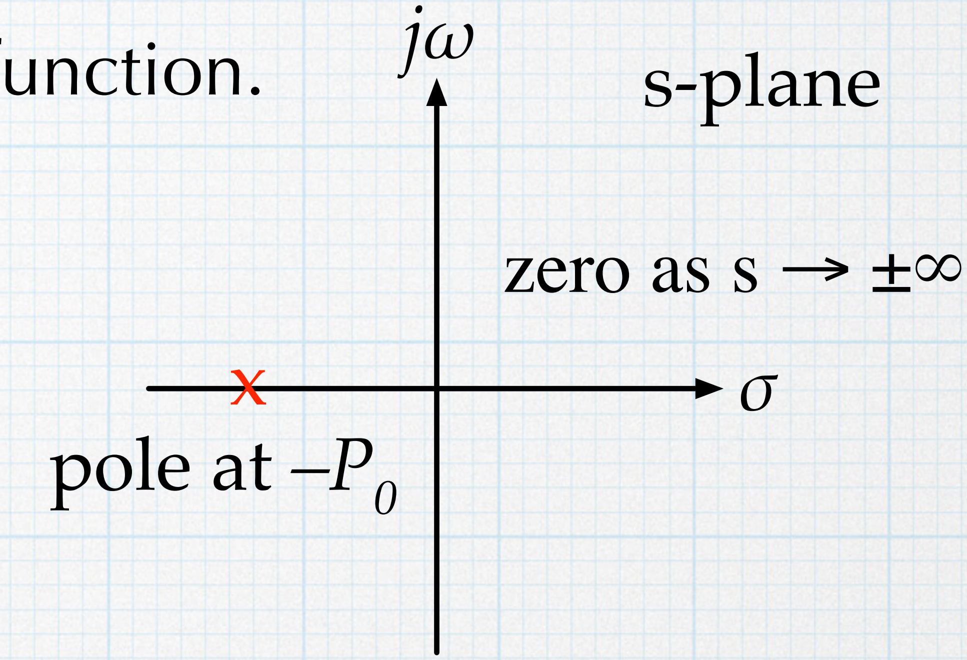
$$T(s) = G_o \cdot \frac{s + Z_o}{s + P_o}$$

There will always be a single pole at $s = -P_o$. The pole must be real (there is only one, so no complex conjugates are not possible) and it must be negative (for stability). There will always be a zero, which can be at $s = 0$, as $s \rightarrow \pm\infty$ (zero at infinity), or somewhere else, $s = -Z_o$. (Note the zero can have a positive value.) There may be a gain factor, G_o, which might be 1 or smaller (for a passive circuit with a voltage divider) or have a magnitude greater than 1 for an active circuit.

The two most important cases are the zero at infinity, which is a low-pass filter and the zero at zero, which is the high-pass filter.

Low-pass

In the case were $a_1 = 0$, we have a low-pass function.



$$T(s) = \frac{a_o}{b_1 s + b_o}$$

In standard form, we write it as:

$$T(s) = G_o \cdot \frac{P_o}{s + P_o}$$

The reason for this form will become clear as we proceed. We will ignore the gain initially and focus on sinusoidal behavior by letting $s = j\omega$.

$$\left. \frac{P_o}{s + P_o} \right|_{s=j\omega} = \frac{P_o}{P_o + j\omega}$$

Re-expressing the complex value in magnitude and phase form:

$$\frac{P_o}{P_o + j\omega} = \left[\frac{P_o}{\sqrt{P_o^2 + \omega^2}} \right] \exp\left(j\theta_{LP}\right) \qquad \theta_{LP} = -\arctan\left(\frac{\omega}{P_o}\right)$$

By looking at the magnitude expression, we can see the low-pass behavior.

For low frequencies ($\omega << P_o$): $$\frac{P_o}{\sqrt{P_o^2 + \omega^2}} \approx \frac{P_o}{\sqrt{P_o^2}} = 1.$$

At high frequencies ($\omega >> P_o$): $$\frac{P_o}{\sqrt{P_o^2 + \omega^2}} \approx \frac{P_o}{\sqrt{\omega^2}} = \frac{P_o}{\omega}.$$

At low frequencies, the magnitude is 1 (the output is equal to the input) and at high frequencies, the magnitude goes down inversely with frequency, consistent with the notion of a low-pass response.

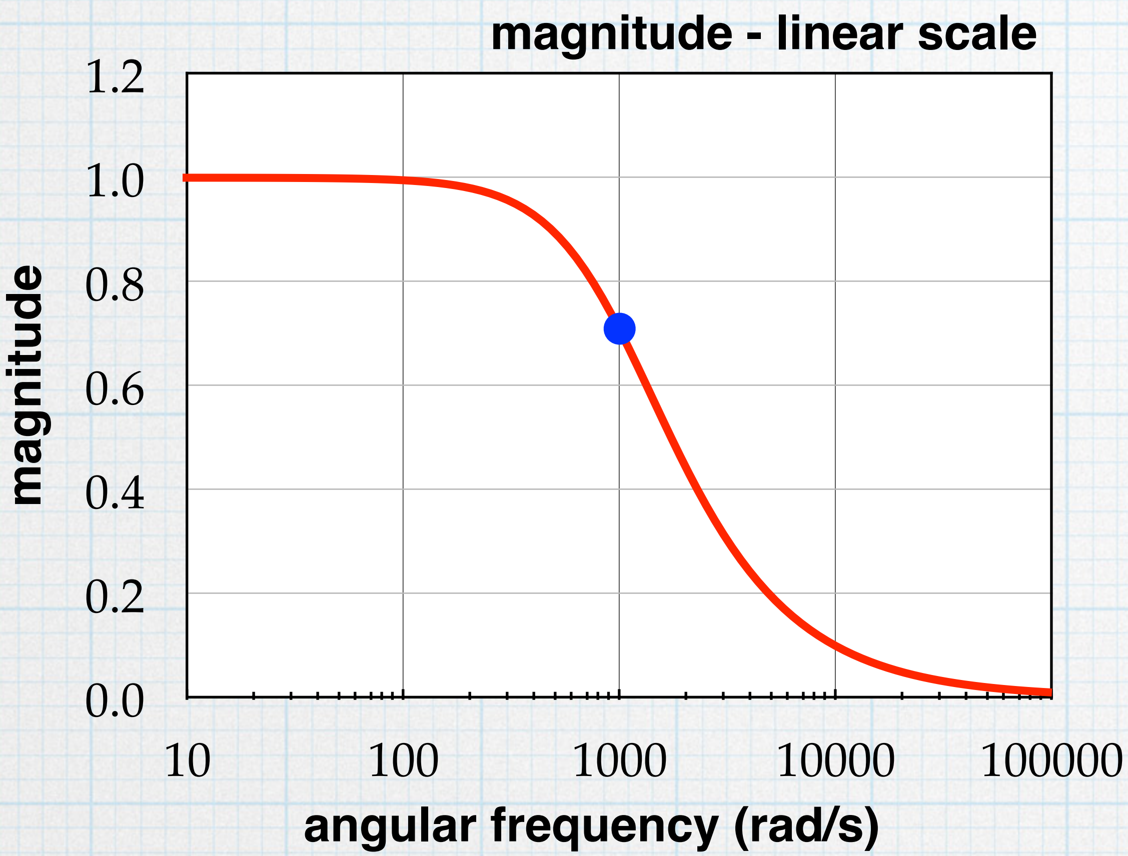
We can also examine the phase at the extremes.

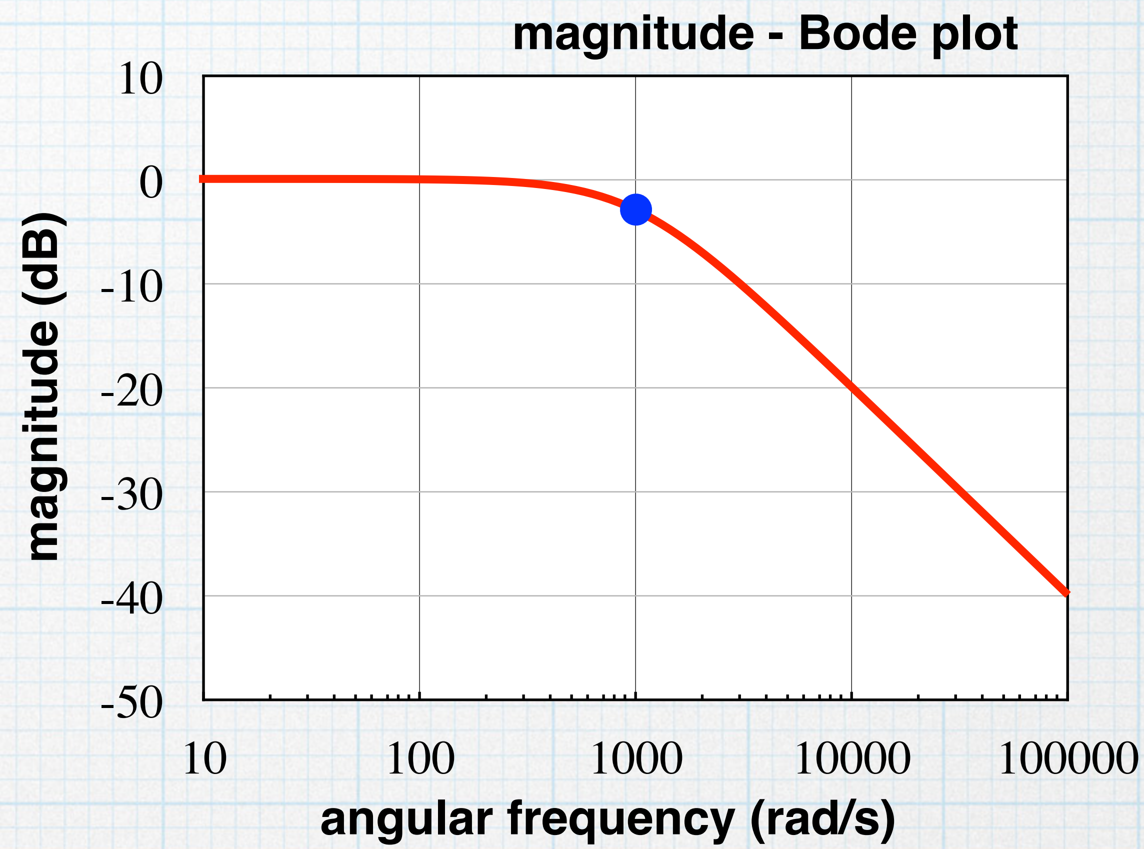
For low frequencies ($\omega \approx 0$): $$\theta_{LP} = -\arctan\left(\frac{\omega}{P_o}\right) \approx 0.$$

At high frequencies ($\omega \to +\infty$): $$\theta_{LP} = -\arctan\left(\frac{\omega}{P_o}\right) \approx -90°.$$

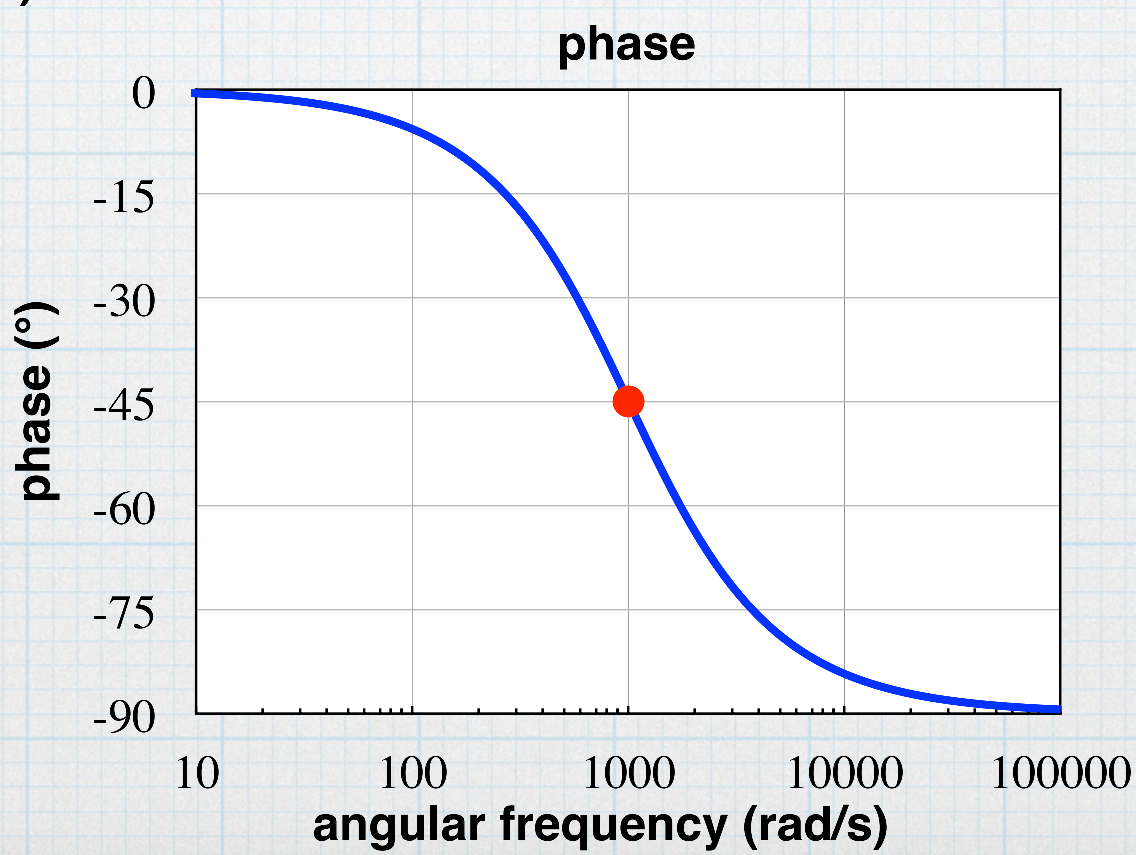
We can use the functions to make the magnitude and phase as frequency response plots.

$$M = \frac{P_o}{\sqrt{P_o^2 + \omega^2}}$$





$$\theta_{LP} = -\arctan\left(\frac{\omega}{P_o}\right)$$



Cut-off frequency

Use the standard definition for cut-off frequency, which is the frequency at which the magnitude is down by $\sqrt{2}$ from the value in the pass-band. For our low-pass function, the pass band is at low frequencies, and the magnitude there is 1. (Again, we are "hiding" G_o by assuming that it is unity. If $G_o \neq 1$, then everything is scaled by G_o.)

$$M = \frac{1}{\sqrt{2}} = \frac{P_o}{\sqrt{P_o^2 + \omega_c^2}}$$

With a bit of algebra, we find that $\omega_c = P_o$. The cut-off frequency is defined by the pole. Tricky! Thus, in all of our equations, we could substitute ω_c for P_o.

We can also calculate the phase at the cut-off frequency.

$$\theta_{LP} = -\arctan\left(\frac{\omega_c}{P_o}\right) = -45^\circ$$

The cut-off frequency points are indicated in the plots on the previous slide.

To emphasize the importance of the corner frequency in the low-pass function, we can express all the previous results using ω_c in place of P_o. On the left are the functions in "standard" form. On the right, the functions are expressed in a slightly different form that is sometimes easier to use.

$$T_{LP}(s) = G_o \cdot \frac{\omega_c}{s + \omega_c} \qquad T_{LP}(s) = \frac{G_o}{1 + \frac{s}{\omega_c}}$$

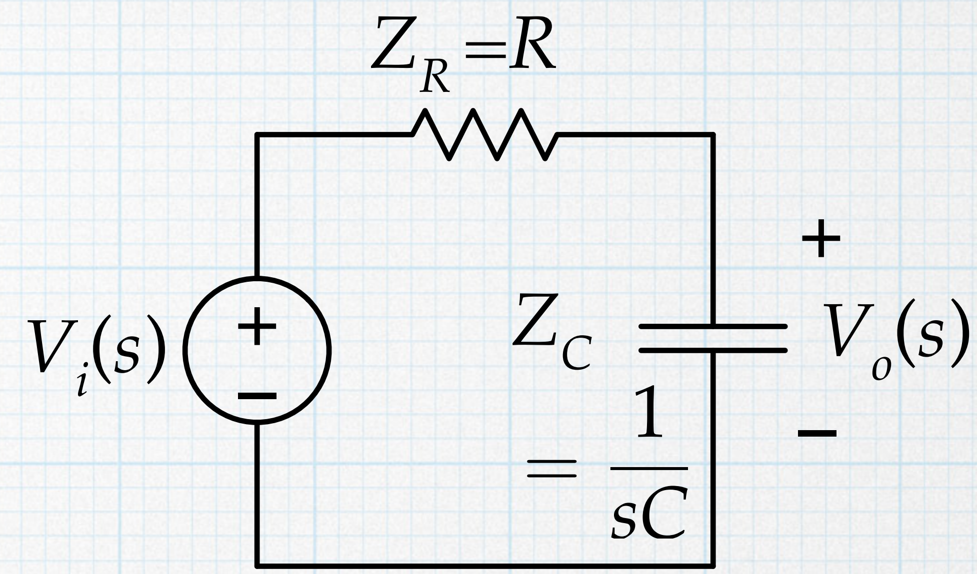
$$T_{LP}(j\omega) = G_o \cdot \frac{\omega_c}{j\omega + \omega_c} \qquad T_{LP}(j\omega) = \frac{G_o}{1 + j\left(\frac{\omega}{\omega_c}\right)}$$

$$\left|T_{LP}(j\omega)\right| = \left|G_o\right| \frac{\omega_c}{\sqrt{\omega^2 + \omega_c^2}} \qquad \left|T_{LP}(j\omega)\right| = \frac{\left|G_o\right|}{\sqrt{1 + \left(\frac{\omega}{\omega_c}\right)^2}}$$

$$\theta_{LP} = -\arctan\left(\frac{\omega}{\omega_c}\right) \qquad \theta_{LP} = -\arctan\left(\frac{\omega}{\omega_c}\right)$$

Again, we could just as easily use real frequency rather than angular frequency. As an exercise: re-express all of the above formulas using f instead of ω.

Low-pass filter circuits: simple RC



Resistor and capacitor in series — output taken across the capacitor.

Use a voltage divider to find the transfer function.

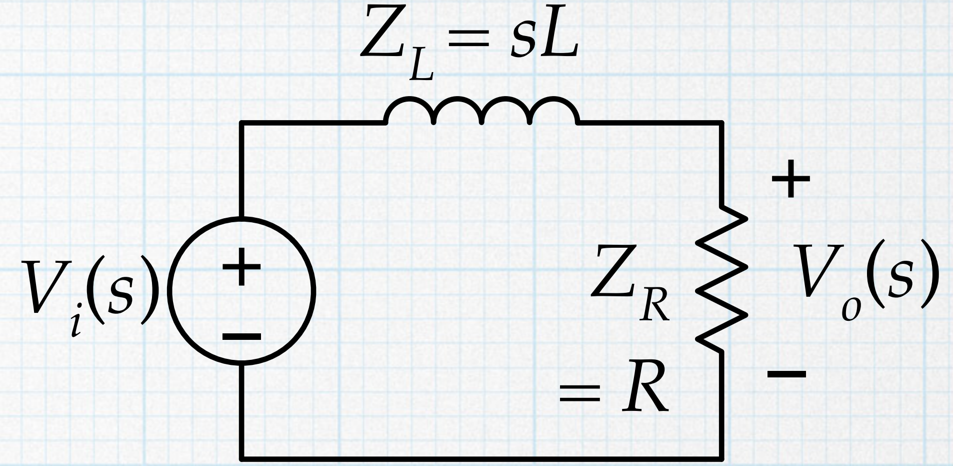
$$V_o(s) = \frac{Z_C}{Z_C + Z_R} V_i(s)$$

$$T(s) = \frac{V_o(s)}{V_i(s)} = \frac{Z_C}{Z_C + Z_R} = \frac{\frac{1}{sC}}{\frac{1}{sC} + R} = \frac{\frac{1}{RC}}{s + \frac{1}{RC}} = \frac{\omega_c}{s + \omega_c}$$

Clearly, this is low-pass with $G_o = 1$ and $\omega_c = \frac{1}{RC}$

The only real design consideration is choosing the *RC* product, which then sets the corner frequency.

Low-pass filter circuits: simple RL



Inductor and resistor in series — output taken across the resistor.

Use a voltage divider to find the transfer function.

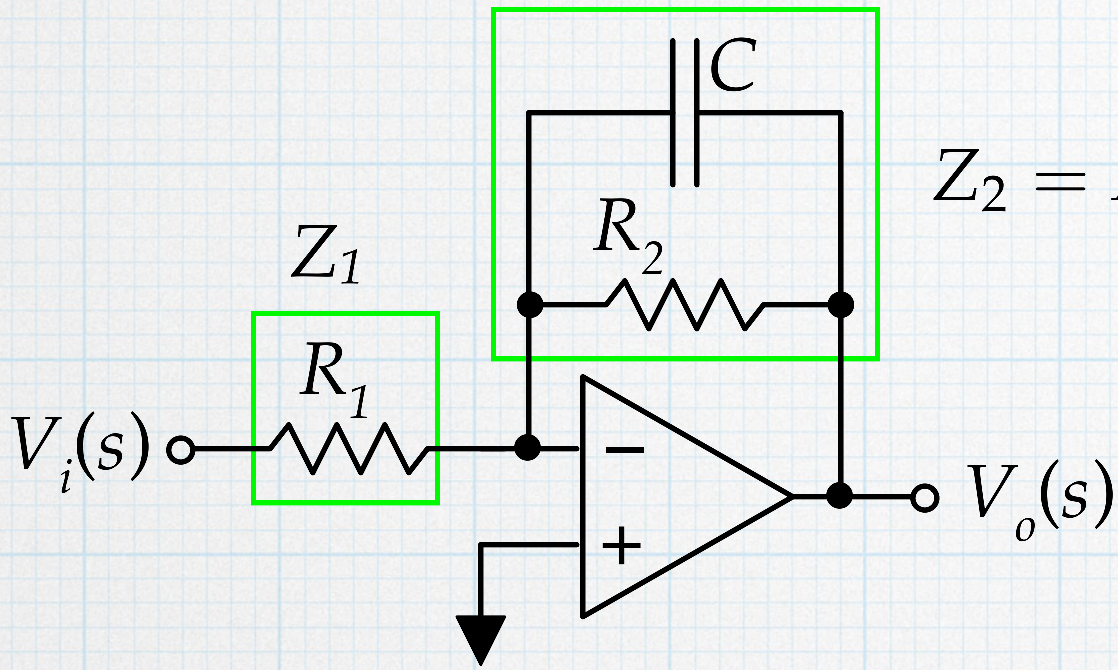
$$V_o(s) = \frac{Z_R}{Z_R + Z_L} V_i(s)$$

$$T(s) = \frac{V_o(s)}{V_i(s)} = \frac{Z_R}{Z_R + Z_L} = \frac{R}{R + sL} = \frac{\frac{R}{L}}{s + \frac{R}{L}} = \frac{\omega_c}{s + \omega_c}$$

Again, low-pass behavior with $G_o = 1$, but now with $\omega_c = \frac{R}{L}$

As with the previous example choosing the "*RL* time constant" , we can define the pass-band of this low-pass filter.

Low-pass filter circuits: inverting op amp



$$Z_2 = R_2 \| \frac{1}{sC} = \frac{R_2}{1 + sR_2C}$$

$$T(s) = \frac{V_o(s)}{V_i(s)} = -\frac{Z_2}{Z_1} = -\frac{\frac{R_2}{1+sR_2C}}{R_1} = \frac{-\frac{R_2}{R_1}}{1 + sR_2C} = \left(-\frac{R_2}{R_1}\right)\frac{\frac{1}{R_2C}}{s + \frac{1}{R_2C}}$$

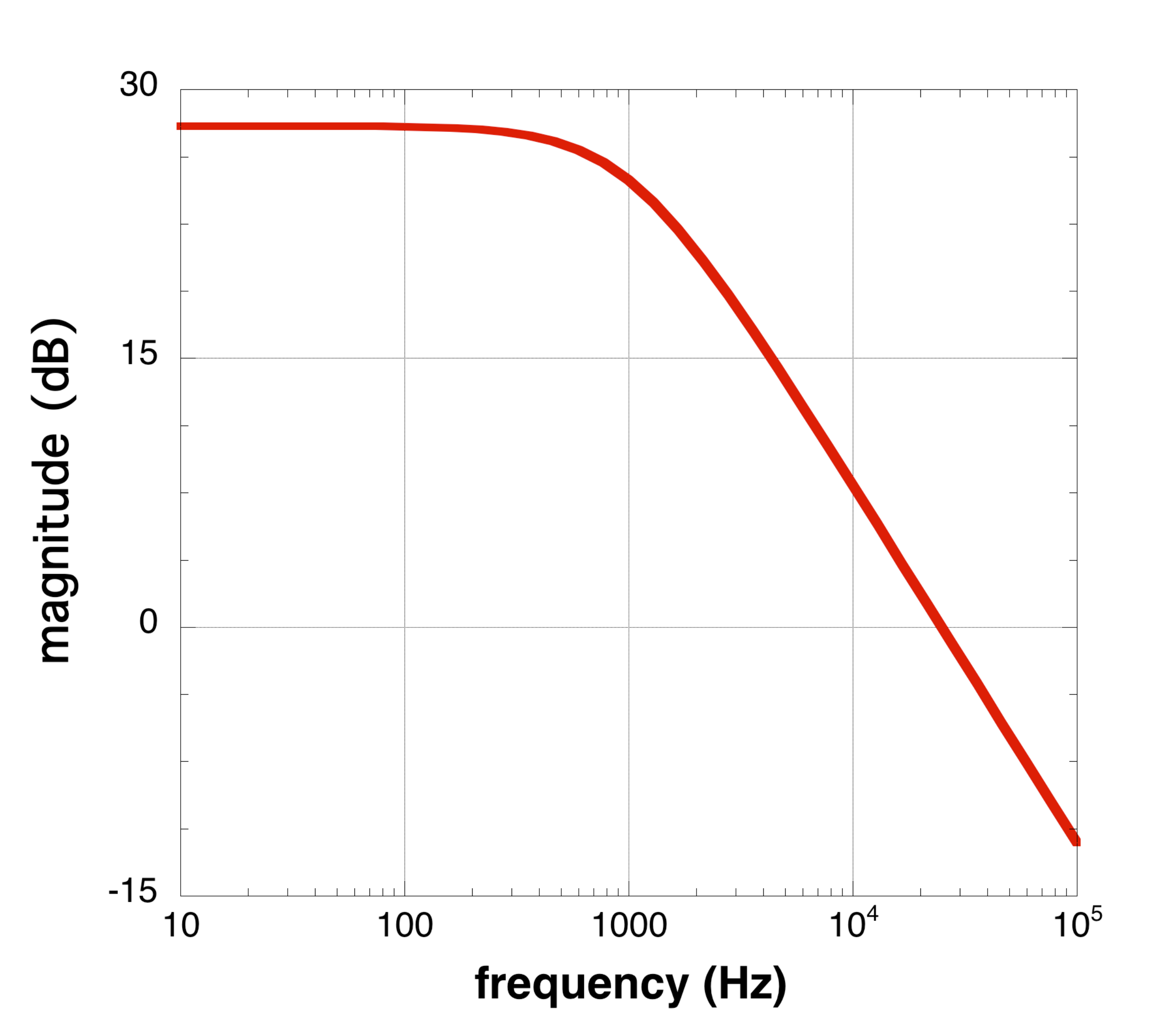
Clearly, this is also low-pass with $G_o = -\frac{R_2}{R_1}$ and $\omega_c = \frac{1}{R_2C}$

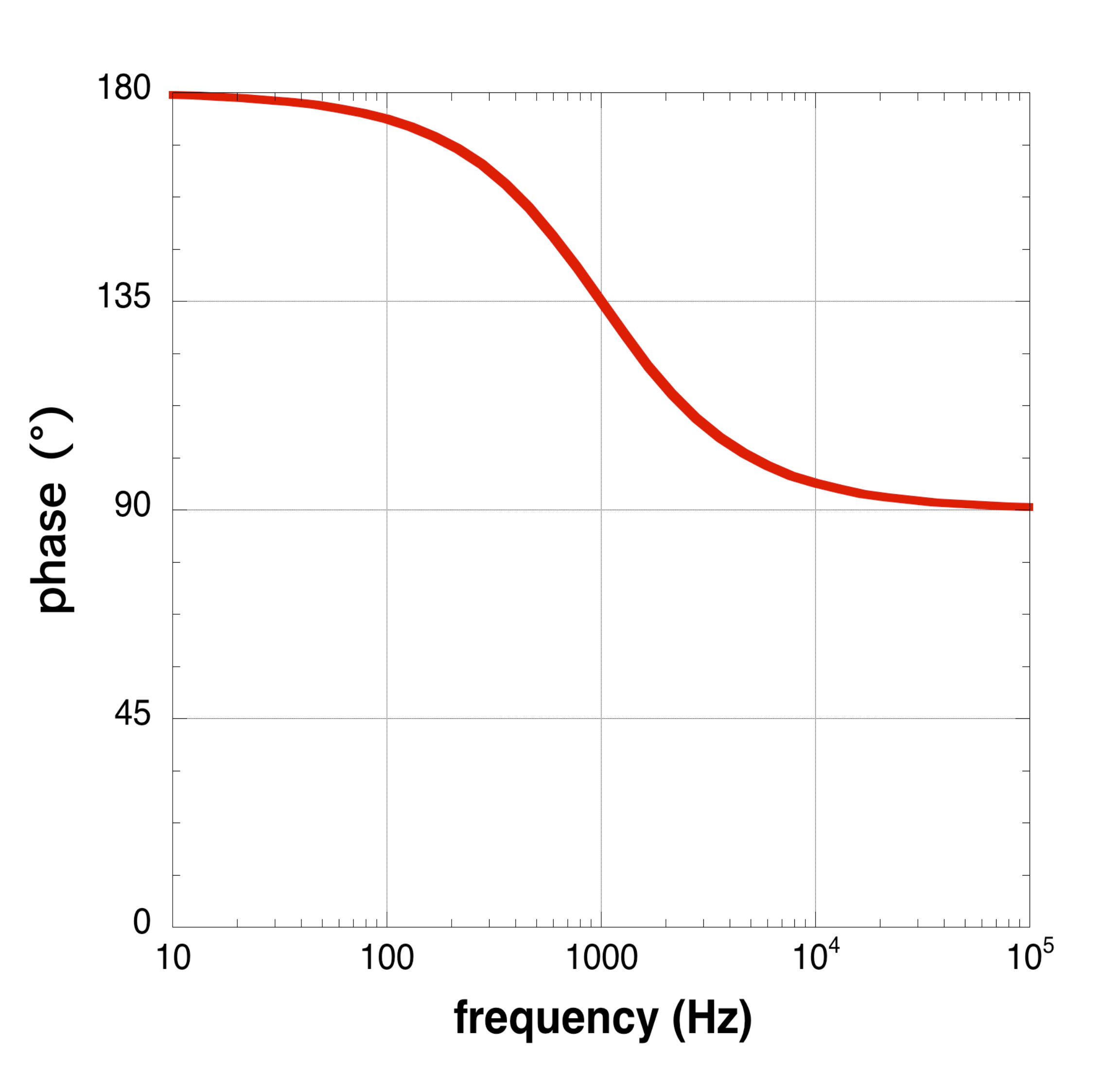
Be careful with the extra negative sign in the gain: –1 = exp(j180°)

At low frequencies: $|T| \rightarrow R_2/R_1$, and $\theta_T \rightarrow$ 180° (= – 180°)

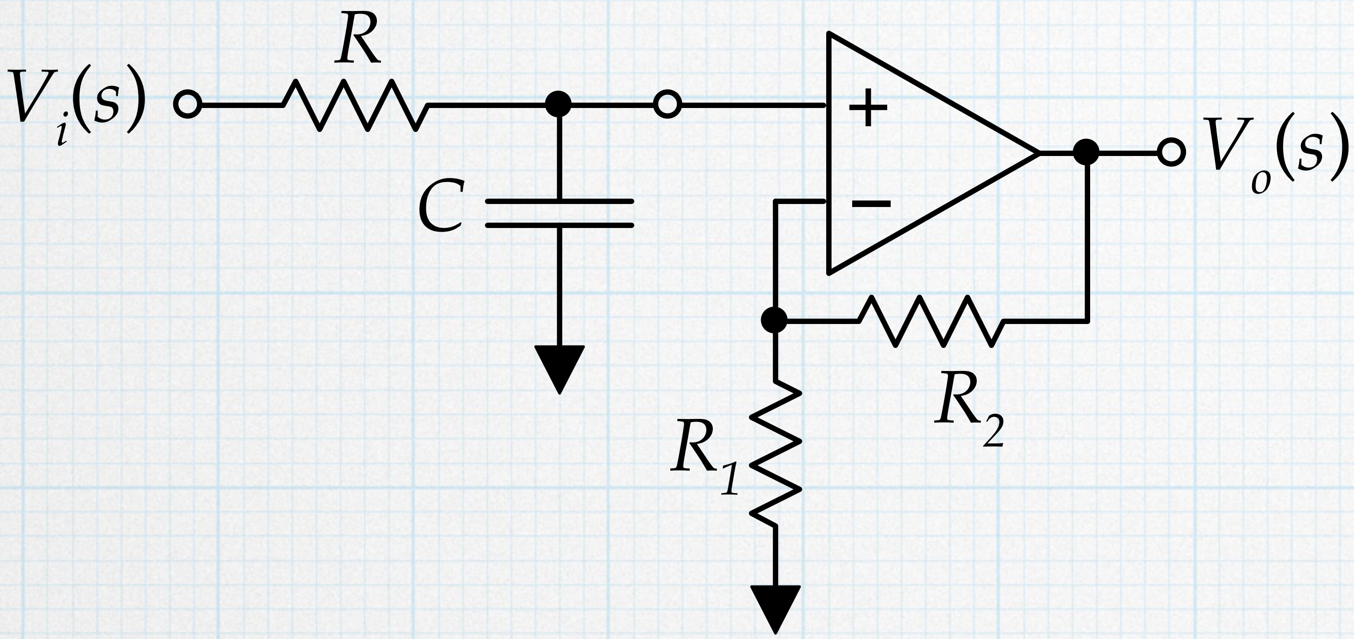
At high frequencies: $|T| \rightarrow (\omega R_1 C)^{-1}$, and $\theta_T \rightarrow$ +90° (= – 270°)

Magnitude and phase plots for an active low-pass filter with R_1 = 1 kΩ, R_2 = 25 kΩ, and C = 6.4 nF, giving f_o = 1000 Hz and G_o = –25 ($|G_o|$ = 28 dB).





Low-pass filter circuits: non-inverting op amp



Note: It might slightly disingenuous to treat this as if it were some new type of filter — we can readily see that it is a simple RC filter cascaded with a simple non-inverting amp. However, it is still a useful circuit.

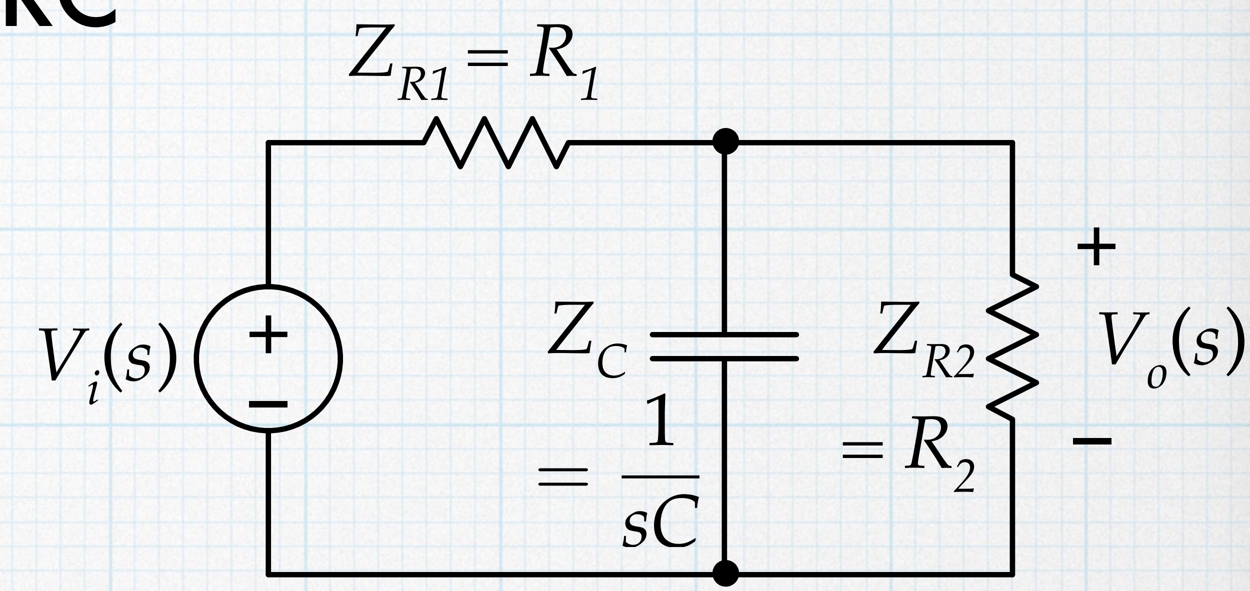
$$V_+(s) = \frac{Z_C}{Z_C + Z_R} = \frac{\frac{1}{RC}}{s + \frac{1}{RC}}$$ simple RC

$$V_o(s) = \left(1 + \frac{R_2}{R_1}\right) V_+(s)$$ non-inverting amp

$$T(s) = \frac{V_o(s)}{V_i(s)} = \left(1 + \frac{R_2}{R_1}\right)\left(\frac{\frac{1}{RC}}{s + \frac{1}{RC}}\right)$$

Low-pass with $G_o = \left(1 + \frac{R_2}{R_1}\right)$ and $\omega_c = \frac{1}{RC}$

Low-pass filter circuits: another RC



$$T(s) = \frac{V_o(s)}{V_i(s)} = \frac{Z_P}{Z_P + Z_{R1}}$$

$$= \frac{\frac{R_2}{1+sR_2C}}{\frac{R_2}{1+sR_2C} + R_1}$$

$$= \frac{R_2}{R_2 + R_1 + sR_1R_2C}$$

$$= \left(\frac{R_2}{R_1 + R_2}\right)\frac{\frac{1}{R_PC}}{s + \frac{1}{R_PC}} \qquad R_P = R_1 \| R_2$$

$$= G_o \cdot \frac{\omega_c}{s + \omega_c}$$

Low-pass.

$$Z_P = Z_{R2} \| Z_C = \frac{R_2\left(\frac{1}{sC}\right)}{R_2 + \frac{1}{sC}} = \frac{R_2}{1 + sR_2C}$$

$$G_o = \frac{R_2}{R_1 + R_2}$$

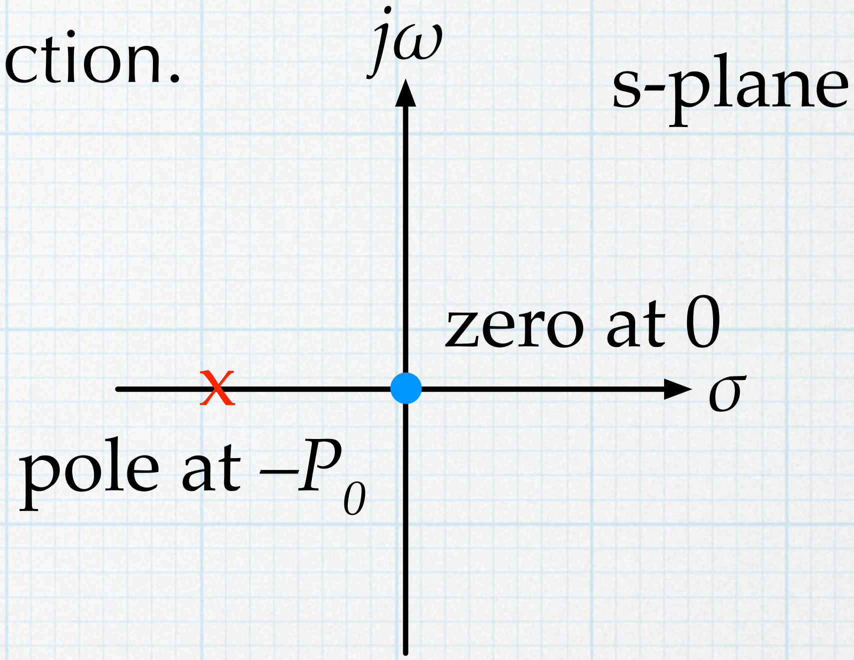
Note the voltage divider. “Gain” < 1.

$$\omega_c = \frac{1}{R_PC}$$

The corner depends on the parallel combination.

High-pass

In the case were $a_0 = 0$, we have a high-pass function.



$$T(s) = \frac{a_1 s}{b_1 s + b_o}$$

In standard form, we write it as:

$$T(s) = G_o \cdot \frac{s}{s + P_o}$$

We will ignore the gain initially (set $G_o = 1$) and focus on sinusoidal behavior by letting $s = j\omega$.

$$\left. \frac{s}{s + P_o} \right|_{s=j\omega} = \frac{j\omega}{P_o + j\omega}$$

Re-expressing the complex value in magnitude and phase form:

$$\frac{j\omega}{P_o + j\omega} = \left[\frac{\omega}{\sqrt{P_o^2 + \omega^2}} \right] \exp\left(j\theta_{HP}\right) \qquad \theta_{HP} = 90° - \arctan\left(\frac{\omega}{P_o}\right)$$

By looking at the magnitude expression, we can see the high-pass behavior.

For low frequencies ($\omega << P_o$): $$\frac{\omega}{\sqrt{P_o^2+\omega^2}} \approx \frac{\omega}{\sqrt{P_o^2}} = \frac{\omega}{P_o}.$$

At high frequencies ($\omega >> P_o$): $$\frac{\omega}{\sqrt{P_o^2+\omega^2}} \approx \frac{\omega}{\sqrt{\omega^2}} = 1.$$

At low frequencies, the magnitude is increasing with frequency, and at high frequencies, the magnitude is 1 (the output is equal to the input). This behavior is consistent with a high-pass response.

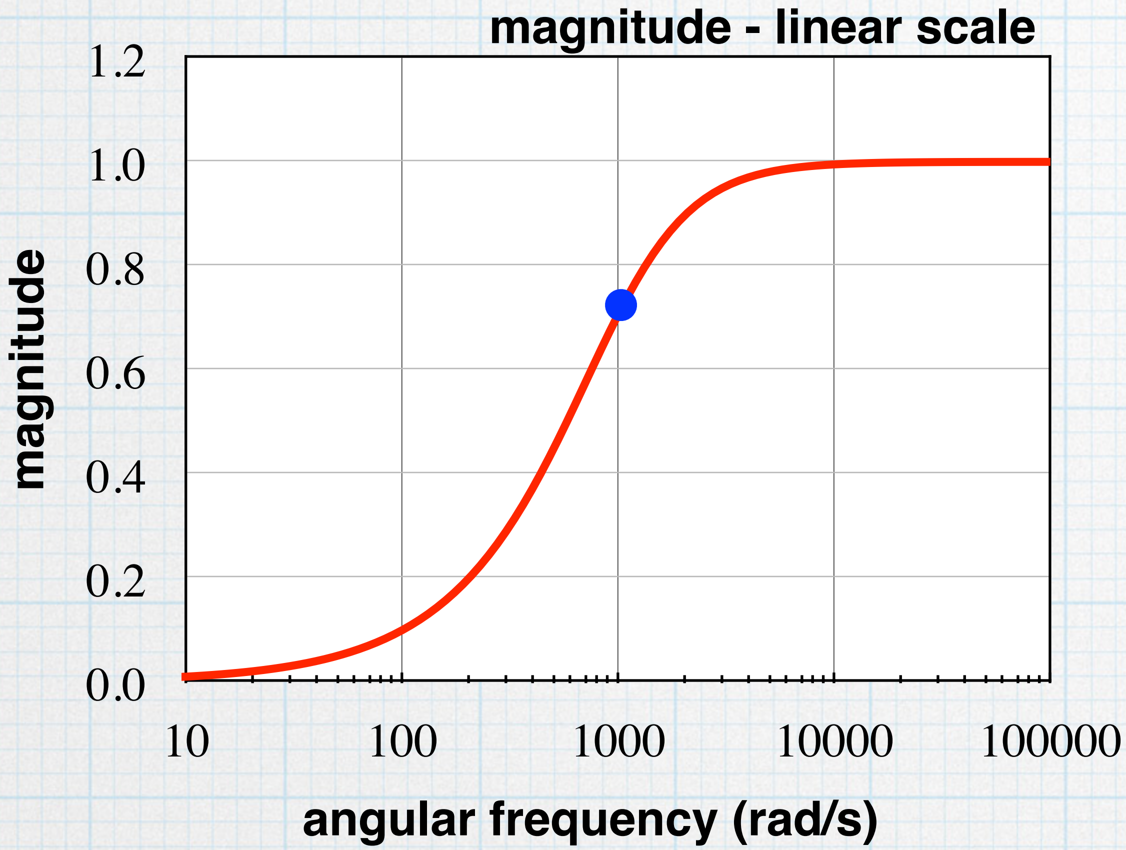
We can also examine the phase at the extremes.

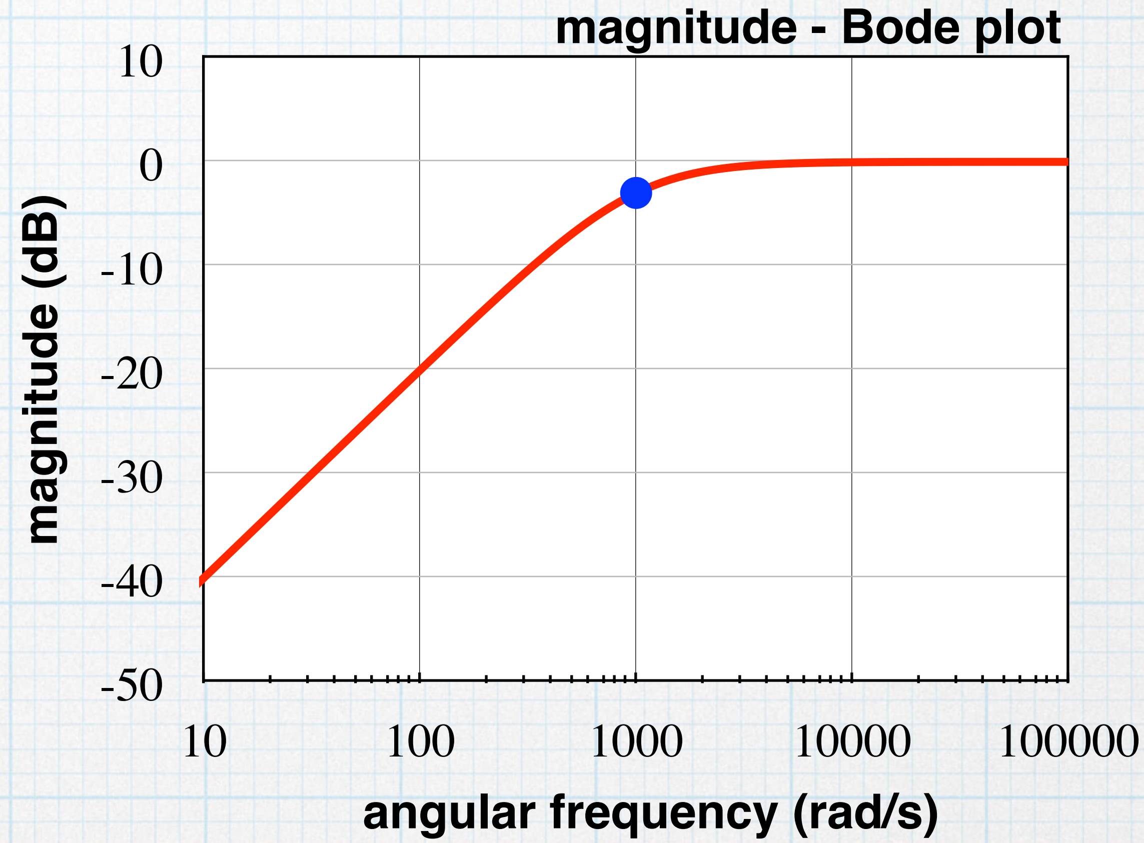
For low frequencies ($\omega << P_o$): $$\theta_{HP} = 90° - \arctan\left(\frac{\omega}{P_o}\right) \approx 90°.$$

At high frequencies ($\omega >> P_o$): $$\theta_{HP} = 90° - \arctan\left(\frac{\omega}{P_o}\right) \approx 0°.$$

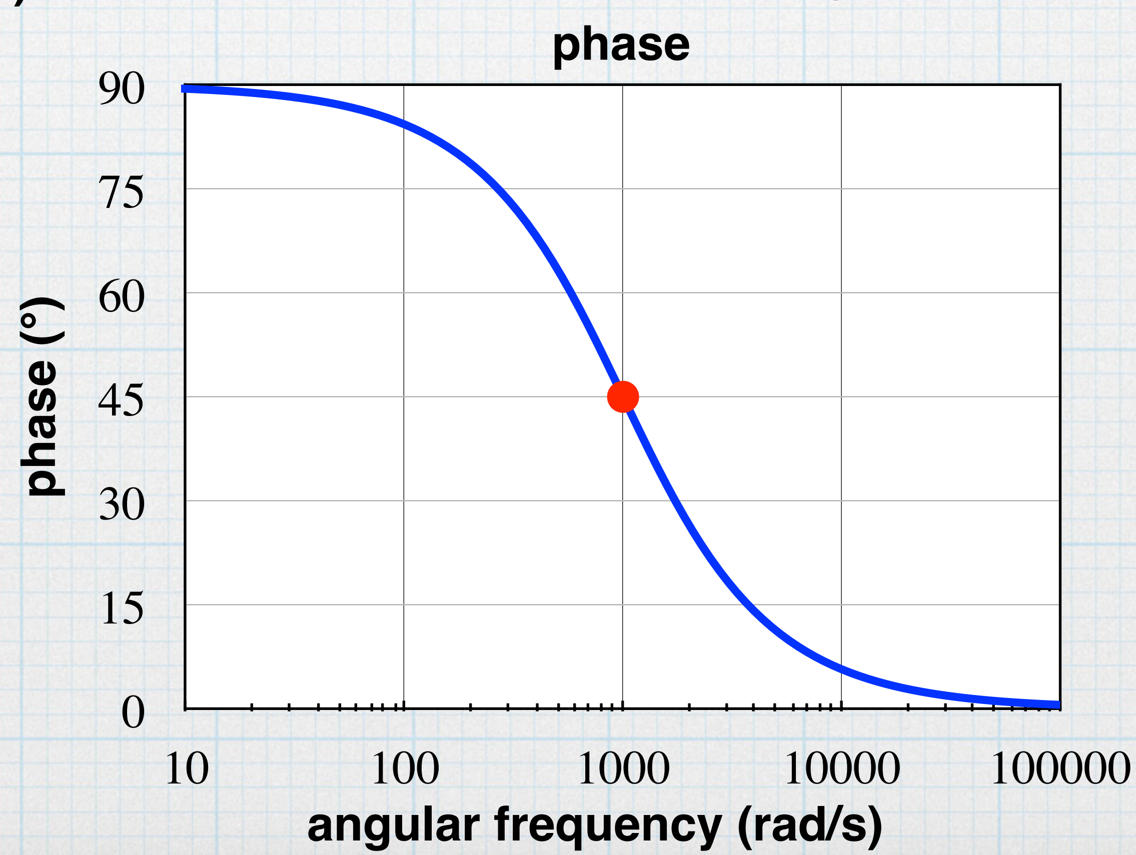
We can use the functions to make the magnitude and phase as frequency response plots.

$$M = \frac{\omega}{\sqrt{P_o^2 + \omega^2}}$$





$$\theta_{HP} = 90^\circ - \arctan\left(\frac{\omega}{P_o}\right)$$



Cut-off frequency

Use the standard definition for cut-off frequency, which is the frequency at which the magnitude is down by $\sqrt{2}$ from the value in the pass-band. For our high-pass function, the pass band is at high frequencies, and the magnitude there is 1. (Again, we are "hiding" G_o by assuming that it is unity. If $G_o \neq 1$, then everything is scaled by G_o.)

$$M = \frac{1}{\sqrt{2}} = \frac{\omega}{\sqrt{P_o^2 + \omega_c^2}}$$

With a bit of algebra, we find that $\omega_c = P_o$. The same result as for low-pass response, except that pass-band is above the cut-off frequency in this case. Once again, we see the importance of the poles in determining the behavior of the transfer functions.

We can calculate the phase at the cut-off frequency.

$$\theta_{HP} = 90^\circ - \arctan\left(\frac{\omega_c}{P_o}\right) = 45^\circ$$

The cut-off frequency points are indicated in the plots on the previous slide.

To emphasize the importance of the corner frequency in the high-pass function, we can express all the previous results using ω_c in place of P_o.

$$\frac{G_o}{\sqrt{2}} = \frac{G_o \cdot P_0}{\sqrt{\omega_c^2 + P_0^2}} \quad \rightarrow \quad P_0 = \omega_c$$

The corner frequency is the value of the pole.

$$T_{HP}(s) = G_o \cdot \frac{s}{s + \omega_c}$$

$$T_{HP}(s) = \frac{G_o}{1 + \frac{\omega_c}{s}}$$

$$T_{HP}(j\omega) = G_o \cdot \frac{j\omega}{j\omega + \omega_c}$$

$$T_{HP}(j\omega) = \frac{G_o}{1 + \left(\frac{\omega_c}{j\omega}\right)}$$

$$= \frac{G_o}{1 - j\left(\frac{\omega_c}{\omega}\right)}$$

$$\left|T_{LP}(j\omega)\right| = G_o \cdot \frac{\omega}{\sqrt{\omega^2 + \omega_c^2}}$$

$$\theta_{HP} = \arctan\left(\frac{\omega}{0}\right) - \arctan\left(\frac{\omega}{\omega_c}\right)$$

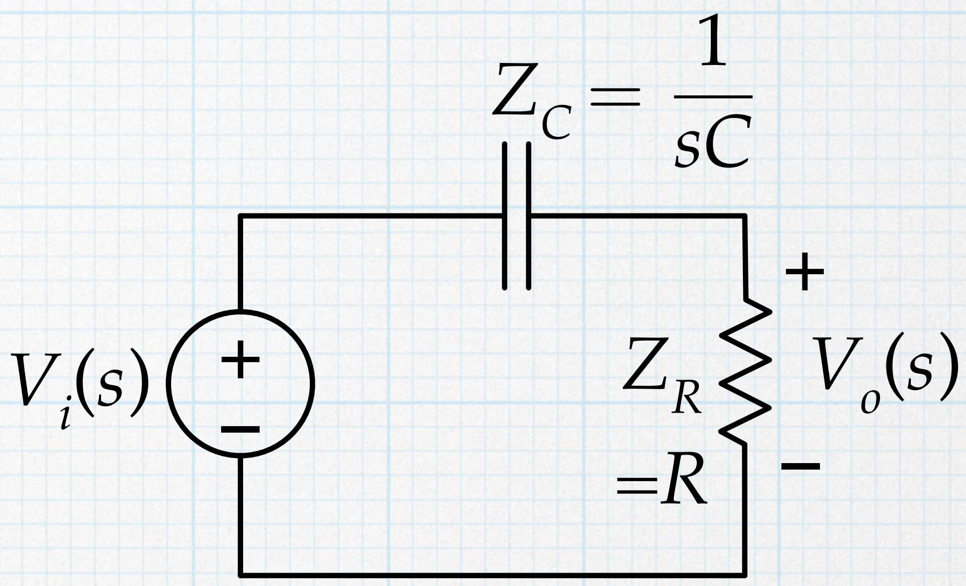
$$= 90^\circ - \arctan\left(\frac{\omega}{\omega_c}\right)$$

$$\left|T_{HP}(j\omega)\right| = \frac{G_o}{\sqrt{1 + \left(\frac{\omega_c}{\omega}\right)^2}}$$

$$\theta_{HP} = +\arctan\left(\frac{\omega_c}{\omega}\right)$$

Exercise: Re-express all of the above formulas using f instead of ω.

High-pass filter circuits: simple RC



Capacitor and resistor in series — output taken across the resistor.

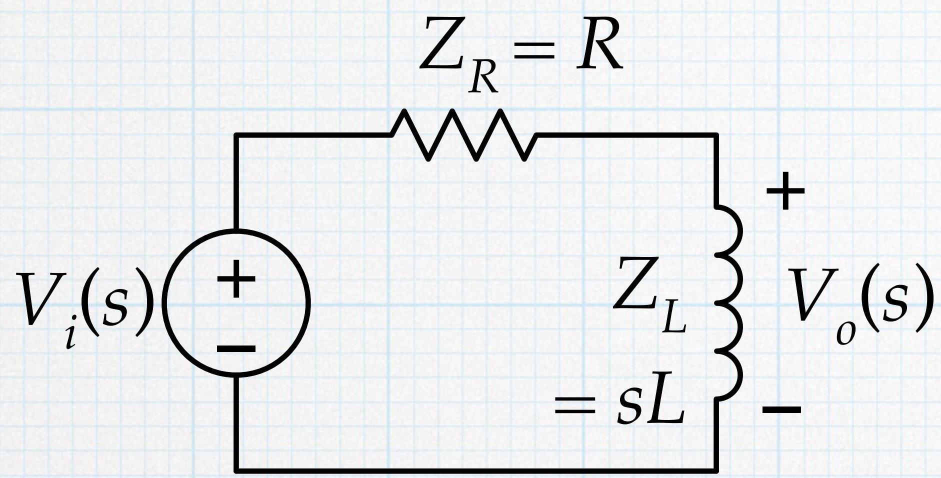
Use a voltage divider to find the transfer function.

$$V_o(s) = \frac{Z_R}{Z_C + Z_R} V_i(s)$$

$$T(s) = \frac{V_o(s)}{V_i(s)} = \frac{Z_R}{Z_R + Z_C} = \frac{R}{R + \frac{1}{sC}} = \frac{s}{s + \frac{1}{RC}} = \frac{s}{s + \omega_c}$$

Clearly, this is high-pass with $G_o = 1$ and $\omega_c = \frac{1}{RC}$

High-pass filter circuits: simple RL

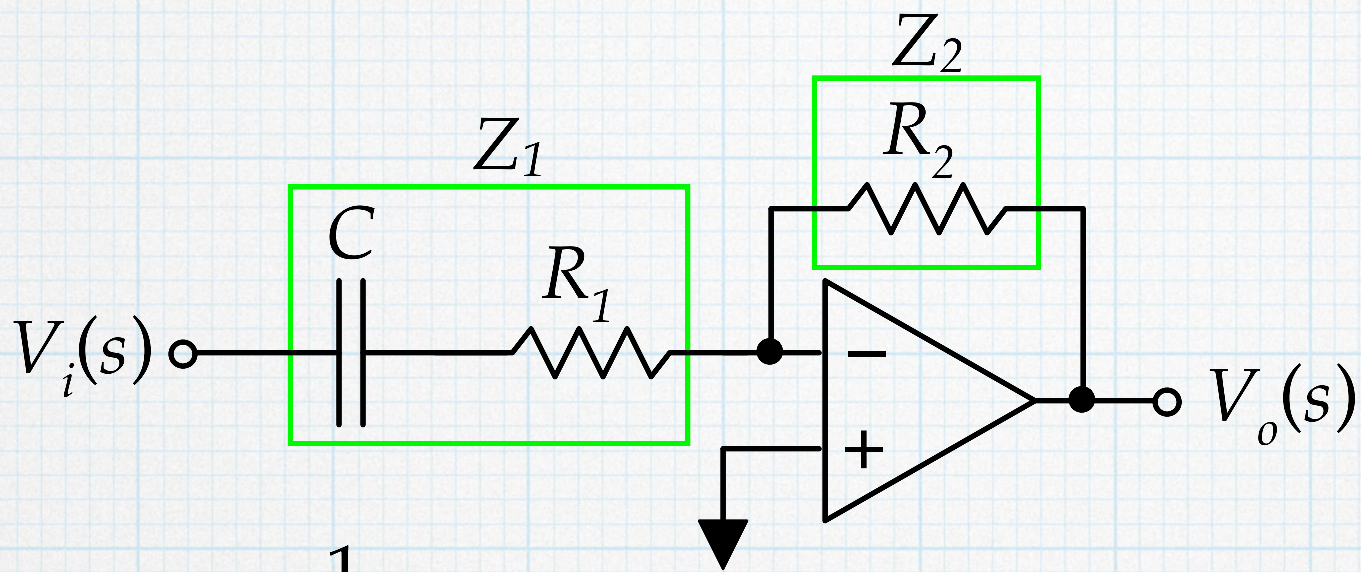


$$V_o(s) = \frac{Z_L}{Z_L + Z_R} V_i(s)$$

$$T(s) = \frac{V_o(s)}{V_i(s)} = \frac{Z_L}{Z_L + Z_R} = \frac{sL}{sL + R} = \frac{s}{s + \frac{R}{L}} = \frac{s}{s + \omega_c}$$

Again, high-pass with $G_o = 1$ but with $\omega_c = \dfrac{R}{L}$

High-pass filter circuits: inverting op amp



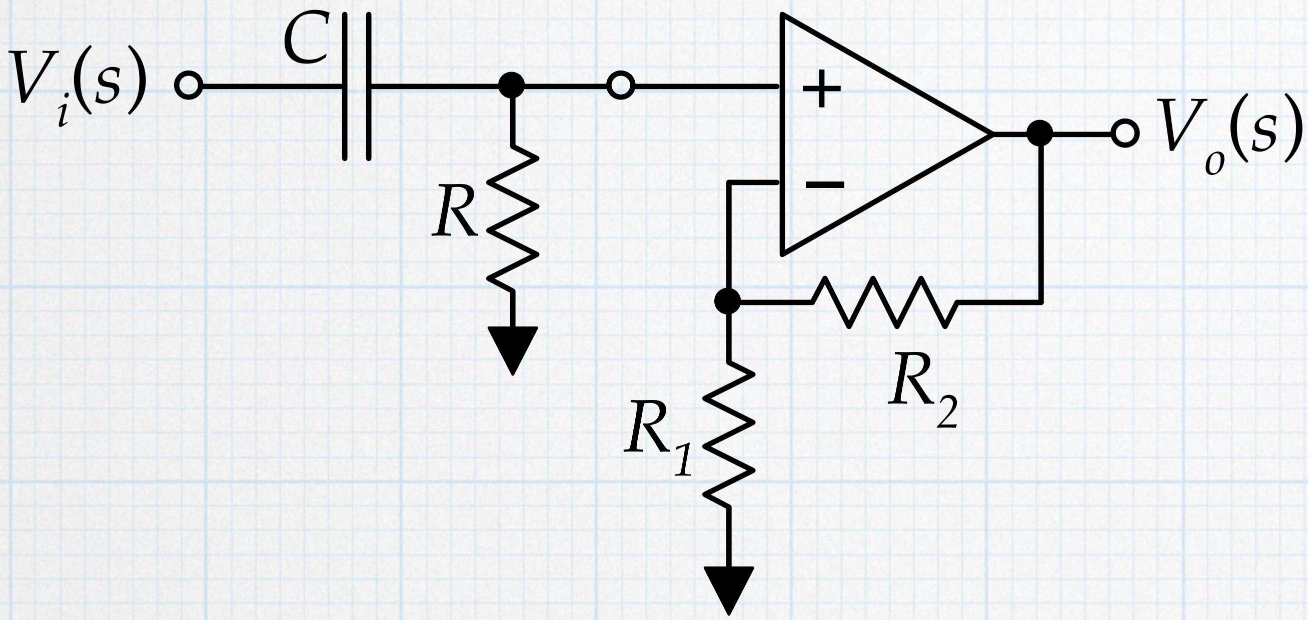
$$Z_1 = R_1 + \frac{1}{sC}$$

$$T(s) = \frac{V_o(s)}{V_i(s)} = -\frac{Z_2}{Z_1} = -\frac{R_2}{R_1 + \frac{1}{sC}} = \left(-\frac{R_2}{R_1}\right)\frac{s}{s + \frac{1}{R_1 C}}$$

We see that this is also high-pass with $G_o = -\frac{R_2}{R_1}$ and $\omega_c = \frac{1}{R_1 C}$

The same comments about the phase apply here: the –1 in the gain factor introduces an extra 180° (or –180°) of phase.

High-pass filter circuits: non-inverting op amp



Again, this is simple *RC* high-pass cascaded with a non-inverting amp.

$$V_+(s) = \frac{Z_R}{Z_C + Z_R} = \frac{s}{s + \frac{1}{RC}}$$

simple *RC* high pass

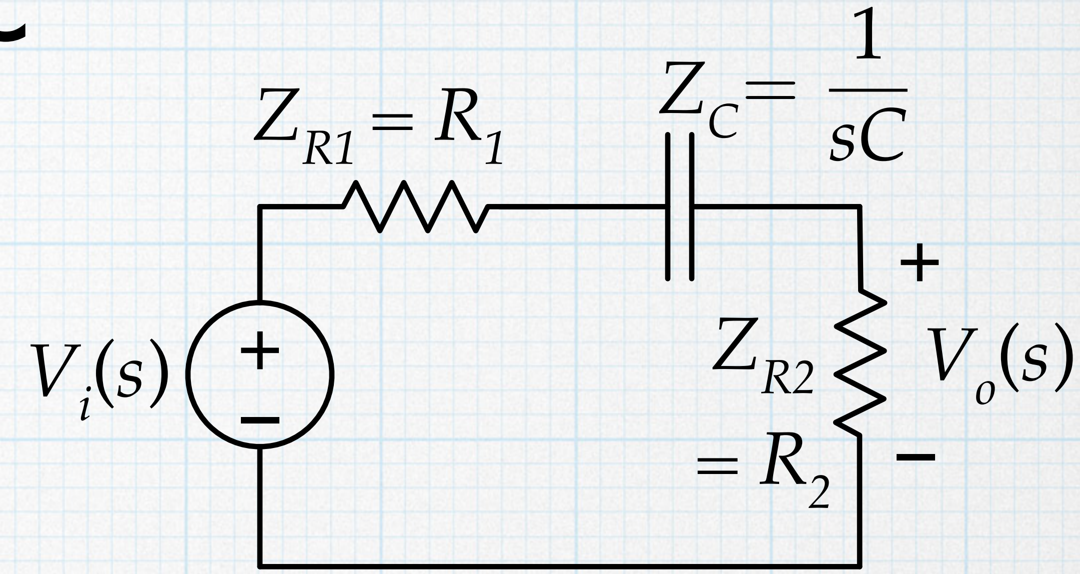
$$V_o(s) = \left(1 + \frac{R_2}{R_1}\right) V_+(s)$$

non-inverting amp

$$T(s) = \frac{V_o(s)}{V_i(s)} = \left(1 + \frac{R_2}{R_1}\right)\left(\frac{s}{s + \frac{1}{RC}}\right)$$

High-pass with $G_o = \left(1 + \frac{R_2}{R_1}\right)$ and $\omega_c = \frac{1}{RC}$

High-pass filter circuits: another RC



$$T(s) = \frac{V_o(s)}{V_i(s)} = \frac{Z_{R2}}{Z_{R2} + Z_{R1} + Z_C}$$

$$= \frac{R_2}{R_2 + R_1 + \frac{1}{sC}}$$

$$= \left(\frac{R_2}{R_1 + R_2}\right) \frac{s}{s + \frac{1}{(R_1+R_2)C}}$$

$$= G_o \cdot \frac{s}{s + \omega_c}$$ High-pass.

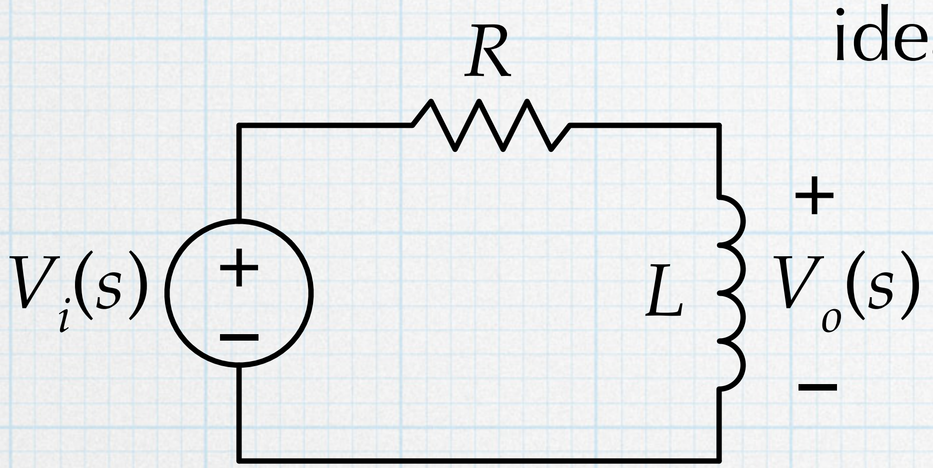
$$G_o = \frac{R_2}{R_1 + R_2}$$ Note the voltage divider. "Gain" < 1.

$$\omega_c = \frac{1}{(R_1 + R_2)\,C}$$ The corner depends on the series combination.

Case study: inductors can be trouble

Cheap inductors can have a relatively large series resistance. This *parasitic* resistance can cause trouble in certain circumstances.

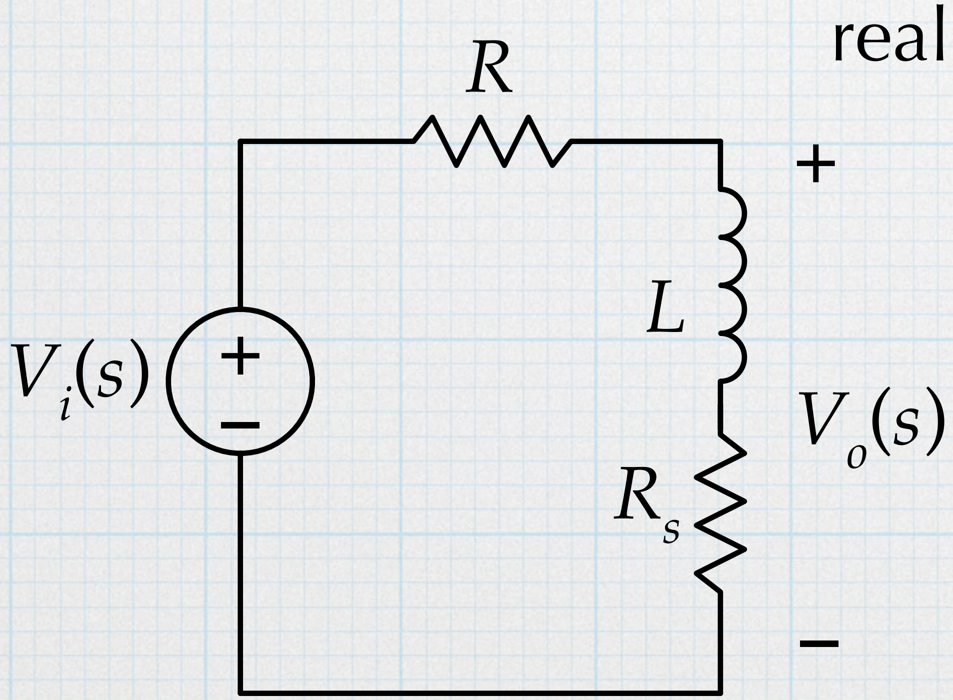
ideal



$$T(s) = \frac{s}{s + \frac{R}{L}} = \frac{s}{s + \omega_o}$$

As usual, zero at $s = 0$, pole at $s = -\omega_c$.

real



$$T(s) = \frac{sL + R_s}{sL + R_s + R_1} = \frac{s + \frac{R_s}{L}}{s + \frac{R_s + R_1}{L}} = \frac{s + \omega_Z}{s + \omega_P}$$

Pole and zero are both shifted!

$$T(j\omega) = \frac{\omega_Z + j\omega}{\omega_P + j\omega} \qquad |T| = \frac{\sqrt{\omega_Z^2 + \omega^2}}{\sqrt{\omega_P^2 + \omega^2}}$$

$$\theta_T = \arctan\left(\frac{\omega}{\omega_Z}\right) - \arctan\left(\frac{\omega}{\omega_P}\right)$$

Effect of inductor parasitic resistance on high-pass filter:

$L = 0.027$ H, $R_1 = 1$ kΩ, and $R_s = 60$ Ω.

The frequency responses for both magnitude and phase are quite different.